One does not have to look far and wide to find visual patterns, but some questions immediately arise.

How can a finite area be enclosed by an infinite perimeter?

The diagrams here are the first four iterations to create a Koch snowflake.

Describe what you observe about:
• the perimeter as you move from the equilateral triangle through to the fourth iteration
• the area enclosed as you move from the triangle to the fourth iteration.

What changes would you expect in the fifth iteration?

How would you physically measure the perimeter at the fifth iteration if the original triangle had sides of 1m in length?

What happens if you start with a square instead of an equilateral triangle?

Can a finite volume enclose an infinite area?

From patterns to generalizations: sequences, series and proof

Concepts
■ Patterns
■ Generalizations

Microconcepts
■ Arithmetic and geometric sequences and series
■ Introduction to limits
■ Sum of series
■ Permutations and combinations
■ Proof
■ Binomial theorem

Can these patterns be explained mathematically?

Can patterns in numbers be useful in real-life situations?

What information would you require to choose the best loan offer? What other scenarios could this be applied to?

If you take out a loan to buy a car how can you determine the actual amount it will cost?

Before you start
You should know how to:

1. Solve linear algebraic equations.
   eg: \( x - 3(x + 5) = 20 - 3x \)
   \( \Rightarrow x - 3x - 15 = 20 - 3x \)
   \( \Rightarrow 2x - 15 = 20 - 3x \)
   \( \Rightarrow x = 35 \)

2. Simplify surds.
   eg: simplify \( \frac{\sqrt{2}}{1 - \sqrt{2}} \)
   \( = \frac{\sqrt{2}(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{\sqrt{2} + 2}{1 - 2} = -2 - \sqrt{2} \)

   eg: simplify \( \frac{x + 3}{x} \cdot \frac{2}{x + 1} \cdot \frac{3x}{x + 1} \)
   \( = \frac{(x + 3)(x) + 2x(x + 1) - 3x^2(x + 1)}{x(x + 1)(x + 1)} \)
   \( = \frac{x^2 + 3x^2 + 2x^2 - 3x^2 - 3x^2}{x^2 + x - 1} \)
   \( = \frac{\cancel{x^2} + x + 3x - 3 + 2x^2 - 2x - 3x^2 - 3x^2}{x(x^2 - 1)} \)
   \( = \frac{-2x^2 + 2x^2 - 3x - 3}{x(x^2 - 1)} \)

Skills check

1. Solve the following equations:
   a. \( 3x + 5(x - 4) = 20 + 4 \)
   b. \( \frac{x + 1}{2x - 1} = \frac{x - 3}{2x + 7} \)

2. Simplify the following:
   a. \( \frac{1 + \sqrt{2}}{1 - \sqrt{2}} \)
   b. \( \frac{\sqrt{2}}{1 - \sqrt{3}} \)

3. Simplify:
   \( \frac{x}{x + 1} \cdot \frac{1}{2x - 1} + \frac{2}{x - 1} \)
1.1 Sequences, series and sigma notation

Opening investigations
You are going to start this chapter by doing some simple arithmetic with the aim of recognizing patterns. The challenge is for you to understand and explain the patterns that emerge. In Investigation 2, you will be asked to propose a conjecture, which is a rule generalizing findings based on observed patterns.

Investigation 1
Work out the following products:

1 × 1
11 × 11
111 × 111
1111 × 1111

1 What pattern do you see emerging?
2 Does this continue as you make the string of 1’s longer?
3 Can you predict when this pattern stops and explain why this happens?

Investigation 2
This diagram represents the floor of a room covered with square tiles. It has a total of nine tiles along the main diagonals (shaded), and five tiles on each side. 25 tiles are used to cover the floor completely.

Another room has a total of 13 square tiles along the diagonals.

1 How many square tiles are there on each side in this other room?
2 How many tiles are needed to completely cover the floor?
3 What if the total number of tiles along the diagonals is 15?
4 What if there is a total of 135 tiles along the diagonals?
5 What if the total number of squares along the diagonals is an even number?
6 Continue to generate data to help you form a conjecture. Can you explain why this rule holds true?
7 How can you write the generalization concisely?
8 Conceptual Why is an algebraic expression more useful than generating numerical values?

A sequence is a list of numbers that is written in a defined order, ascending or descending, following a specific rule. Each of the numbers making up a sequence is called a term of that sequence. Sometimes a sequence is also referred to as a progression.

Look at the following sequences of numbers and identify the rule which would help you obtain the next term.

i 7, 5, 3, 1, ...
ii 2, 4, 8, 16, ...
iii 1, 3, 9, 27, ...

Sequences may be finite or infinite.

The sequence 7, 5, 3, 1, −1, −3 is a finite sequence with six terms, whereas the sequence 7, 5, 3, 1, −1, −3, ... is an infinite sequence with an infinite number of terms. The distinction is indicated by the ellipsis (...) at the end of the sequence.

A sequence is sometimes written in terms of the general term as \(\{u_r\}\), where \(r\) can take values 1, 2, 3, ...

If the sequence is finite then \(r\) will terminate at some point.

The sequence \(\{u_r\} = (3r - 1)\), where \(r \in \mathbb{Z^+}\) represents the infinite sequence 2, 5, 8, 11, ..., whereas the sequence \(\{u_r\} = \left\{\frac{1}{r}\right\}\) where \(r \in \mathbb{Z^+}\), \(r \leq 5\) represents the finite sequence 1, \(\frac{1}{2}\), \(\frac{1}{3}\), \(\frac{1}{4}\), \(\frac{1}{5}\).

All the terms in a sequence added together are called a series. Like sequences, series can be finite or infinite.

The series obtained by adding the six terms of the sequence 7, 5, 3, 1, −1, −3 is \(7 + 5 + 3 + 1 - 1 - 3 = 12\). This is a finite series. The sum \(1 + 3 + 9 + 27 + 81 + \ldots\) continues indefinitely and is an infinite series.

The set of positive integers \(\mathbb{Z^+}\) can be written as \{1, 2, 3, 4, 5, ..., \(r\), ...\}, where the letter \(r\) is used to represent the general term. If the positive integers which are multiples of 5 are considered, then the set \{5, 10, 15, 20, ..., 5\(r\), ...\} is obtained. In this case the general term is 5\(r\) where \(r\) is any positive integer. The harmonic series is the infinite sum of the reciprocals of positive integers, ie \(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r} + \ldots\)

Series can be represented in compact form using sigma (\(\sum\)) notation. This makes use of the general term written in terms of \(r\), which often represents a positive integer.

The sum of the first 10 positive integers can be written as follows using sigma notation:

\[
\sum_{r=1}^{10} r
\]

The largest value that \(r\) can take

The smallest value that \(r\) can take

Read this as “The sum of \(r\) from \(r = 1\) to \(r = 10\).”

If you want to write the sum of the positive multiples of 5 less than 100, then you first need to think of the general term, which is 5\(r\), and then establish the range of values that \(r\) can take. The smallest positive
Example 1
For each of the following sequences, write the next three terms and find the general term:

a) 2, 7, 12, 17, ...

The next three terms of this sequence are 22, 27, 32.
The sequence can be written as: $a_n = 5n + 2$.

b) 2, 6, 12, 20, ...

The next three terms are 30, 42, 56.
The sequence can be written as: $a_n = 2n + 4$.

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \ldots$

The next three terms are $\frac{4}{7}, \frac{5}{11}, \frac{6}{19}$.
The sequence can be written as: $a_n = \frac{n}{2n + 1}$.

d) 5, 10, 20, 40, ...

The next three terms are 80, 160, 320.
The general term is $5 \times 2^{n-1}$, where $n$ can take the values 1, 2, 3, ...

Example 2
Write down the first three terms of each of the following sequences:

a) $u_n = (5n - 2)$, $n \in \mathbb{Z}$

$u_1 = 3, u_2 = 7, u_3 = 11$

b) $u_n = \left\{ \frac{n^2}{2}, \text{ if } n \text{ is even} \right\}$, $n \in \mathbb{Z}$

$u_1 = \frac{1}{2}, u_2 = 1, u_3 = \frac{3}{2}$

Example 3
Write each of the following sequences using the general term:

a) 3, 6, 9, 12, ...

$u_n = 3n$, $n \in \mathbb{N}$

b) 2, -10, 50, -250

$u_n = 2(-5)^{n-1}$, $n \leq 4$

c) $\frac{1}{3}, \frac{2}{5}, \frac{3}{9}, \ldots$

$u_n = \left\{ \frac{r}{2r + 1}, \text{ if } r \in \mathbb{Z} \right\}$

This is an infinite sequence of the positive multiples of 3.

This finite sequence can be written as: $2, 2 \times (-5), 2 \times 25, 2 \times (-125)$ which can be rewritten in terms of powers of -5:

$= 2 \times (-5)^0, 2 \times (-5)^1, 2 \times (-5)^2, 2 \times (-5)^3$

In this infinite sequence, the numerators are the positive integers and the denominators are successive odd integers greater than 1.
Example 4 shows how to expand a series written in sigma notation.

**Example 4**

For each of the following series written in sigma notation, write the first five terms:

- **a** \[ \sum_{r=1}^{6} (r-1) \]
- **b** \[ \sum_{r=1}^{6} (-1)^r r \]
- **c** \[ \sum_{r=1}^{6} \frac{r+1}{2r-1} \]

Substitute \( r = 1 \) to 5 for the first through to the fifth term. Simplify.

\[ \begin{align*}
   \text{a} & \quad = 1 \times 0 + 2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + \ldots + 6 \times 5 \\
   & \quad = 0 + 2 + 6 + 12 + 20 + \ldots + 90 \\

   \text{b} & \quad = \sum_{r=1}^{6} (-1)^r r \\
   & \quad = (-1)^1 \times 1^2 + (-1)^2 \times 2^2 + (-1)^3 \times 3^2 + (-1)^4 \times 4^2 + (-1)^5 \times 5^2 + \ldots \\
   & \quad = -1 + 4 - 9 + 16 - 25 + \ldots \\

   \text{c} & \quad = \sum_{r=1}^{6} \frac{r+1}{2r-1} \\
   & \quad = \frac{1+1}{2-1} + \frac{2+1}{4-1} + \frac{3+1}{6-1} + \frac{4+1}{8-1} + \frac{5+1}{10-1} \\
   & \quad = 2 + \frac{3}{5} + \frac{4}{7} + \frac{5}{9} + \ldots 
\end{align*} \]

In Example 5 you will see how a given series can be written in sigma notation.

**Example 5**

Write each of the following series in sigma notation:

- **a** \[ 3 + 11 + 19 + 27 + 35 \]
- **b** \[ -1 - 1 + 1 - 1 + 1 + \ldots \]
- **c** \[ -6 + 12 - 24 + 48 - 96 + 192 \]

**Exercise 1A**

1 For each of the following sequences, write the next three terms and find the general term:

- **a** \( 3, 4, 5, 6, 7, 5, \ldots \) \\
- **b** \( 8, 14, 22, 30, \ldots \) \\
- **c** \( 5, 9, 27, 81, \ldots \) \\
- **d** \( \frac{4}{7}, \frac{7}{10}, \frac{10}{13}, \ldots \)

2 Write down the first five terms of each of the following sequences:

- **a** \( u_r = 3 - 2r \) \\
- **b** \( u_r = \frac{-r}{2r+1} \) \\
- **c** \( u_r = 2r + (-1)^r \) \\
- **d** \( u_r = (-1)^r \times 2 \) \\
- **e** \( v_r = \frac{3}{2^r} \)

3 Write each of the following sequences using the general term:

- **a** \( 5, 10, 15, 20, \ldots \) \\
- **b** \( 6, 14, 22, 30, \ldots \) \\
- **c** \( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots \) \\
- **d** \( \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \) \\
- **e** \( 0, 3, 8, 15, \ldots \)

4 Write each of the following series in full:

- **a** \[ \sum_{r=1}^{5} 2r(3-r) \] \\
- **b** \[ \sum_{r=1}^{5} (-1)^r r^2 \]

**Developing inquiry skills**

Now go back to the opening question. Suppose the length of each side of the first triangle is 81 cm. Can you work out the length of each side of the figure in each iteration? Tabulate your results and try to find a pattern and then make a conjecture.
1.2 Arithmetic and geometric sequences and series

Investigation 3
Whenever you go through airport security, you have to place your hand luggage, coat, phone, etc on a conveyer belt which then takes it through an x-ray scanner.

When answering the following questions, you can assume the following:
- Trays are placed on the conveyer belt with no gaps between them.
- The length of each tray is 60 cm.
- The conveyer belt is moving at 10 cm per second.
- Each person uses three trays.

1. Copy and complete the following table:

<table>
<thead>
<tr>
<th>Number of people ahead of you</th>
<th>Distance of your first tray to machine, d (m)</th>
<th>Waiting time, T (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.8</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>2.6</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>n</td>
<td>...</td>
</tr>
</tbody>
</table>

2. What patterns do you see emerging?
3. Now assume that there is a 30 cm gap separating trays belonging to different passengers. Construct and complete a table similar to the one above.
4. How have the patterns changed?
5. What happens if the distance between the trays of individual passengers changes to 50 cm? 60 cm? 80 cm?
6. How have the patterns changed?

7. **Factual** What do you notice about consecutive terms in the second and third columns?
8. **Factual** How would you generalize the relationship between the distance from the machine to your first tray and the number of people ahead of you?
9. **Factual** Write down the relationship between the waiting time and the number of people ahead of you.
10. **Conceptual** How did the recognition of patterns help you make predictions about waiting times?

Arithmetic sequences and series

A growth pattern that is represented by a linear relationship is also known as an arithmetic sequence, which is defined as follows:

If the difference between two consecutive numbers in a sequence is constant then it is an arithmetic sequence or an arithmetic progression. The constant difference is called the common difference and is denoted by $d$.

Consider how an arithmetic sequence with first term $u_1$ and common difference $d$ grows:
- First term $u_1$
- Second term $u_2 = u_1 + d$
- Third term $u_3 = u_2 + d = u_1 + 2d$
- Fourth term $u_4 = u_3 + d = u_1 + 3d$

This leads to the general term $u_n = u_1 + (n-1)d$.

An arithmetic sequence with first term $u_1$ and common difference $d$ has general term $u_n = u_1 + (n-1)d$.

The next four examples show you how to use the general term formula to answer different types of questions.

**Example 6**
The fourth term of an arithmetic sequence is 18 and the common difference is $-5$. Determine the first term and the $n$th term.

$u_4 = u_1 + 3(-5) = 18$
$\Rightarrow u_1 = 18 + 15 = 33$
$u_n = 33 + (n-1)(-5)$
$\Rightarrow u_n = 38 - 5n$

Using $u_n = u_1 + (n-1)d$.

**Example 7**
Find the number of terms in the following arithmetic sequences:
- a) $20, 23, 26, \ldots, 83$
- b) $34, 30, 26, \ldots, -30$
- c) $6a, 4a, 2a, \ldots, -22a$
- d) $23, -20, 3$

**Example 6**

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Number of terms</th>
</tr>
</thead>
</table>
| a) $20, 23, 26, \ldots, 83$ | $u_1 = 20$, $d = 3$
|   | $u_n = 17 + 3(n-1)$
|   | $\Rightarrow n = 22$ |
| b) $34, 30, 26, \ldots, -30$ | $u_1 = 34$, $d = -4$
|   | $u_n = 38 - 4(n-1)$
|   | $\Rightarrow n = 17$ |
| c) $6a, 4a, 2a, \ldots, -22a$ | $u_1 = 6a$, $d = -2a$
|   | $u_n = 8a - 2an$|
|   | $\Rightarrow n = 15$ |
| d) $23, -20, 3$ | $d = 23 - 20 = 3$
|   | Using $u_n = u_1 + (n-1)d$ |
|   | Solve the linear equation to obtain $n$. |

**HINT**
A recursive equation is one in which the next term is defined as a function of earlier terms. In the case of an arithmetic sequence the recursive equation is $u_n = u_{n-1} + d$. 

Number and algebra
Example 8

Three numbers are consecutive terms of an arithmetic sequence. The sum of the three numbers is 45, and their product is 3240. Find the three numbers.

Let the three numbers be \(u, u + d\), and \(u + 2d\).

Taking the sum of the numbers.

\[u + (u + d) + (u + 2d) = 45\]

Solving simultaneously.

\[3u + 3d = 45\]

\[u = 15\]

Substitute \(u = 15\) and divide by 15.

\[d = \pm 3\]

The three numbers are 12, 15 and 18.

Example 9

The second term of an arithmetic sequence is 20 and the seventh term is 55. Find the first term and the common difference of the sequence.

\[u_2 = u_1 + d = 20\]

\[u_7 = u_1 + 6d = 55\]

\[d = 7\]

\[u_1 = 20 - 7 = 13\]

Or

\[u_5 = u_1 + 5d = 55 - 20\]

\[u_1 = 20 - 7 = 13\]

Solving simultaneously.

\[u_n = u_1 + (n - 1)d\]

Write \(u_n\) in terms of \(u_1\).

Solve for \(d\). \n
The sum of an arithmetic sequence

Investigation 4

Miss Sandra, the grade 5 teacher, pairs up her students and gives each pair 55 cards numbered from 1 to 55. She tells the students that she wants them to use these cards to find the sum of the numbers 1 + 2 + 3 + ... + 55.

Michela and Grisha start by laying out the cards in ascending order. Michela takes away the first card and the last card and notes that their sum is 56. Grisha then takes the first and last card from the cards that remain and notes that their sum is also 56. They continue to do this until just one card is left.

1. Which card will this be?
2. Using the information above, how would you determine the sum of the first 55 positive integers?
3. What if you wanted to find the sum of the first 1000 positive integers?
4. Explain the importance of the actual number of terms added.
5. Repeat the process for finding the sum of:
   a. the first 100 even numbers
   b. the positive multiples of 3 less than 1000.
6. How was Michela’s and Grisha’s method more efficient?

Reflect on Investigation 6 and explain how the method used is equivalent to the direct derivation for the sum of an arithmetic series containing \(n\) terms, with first term \(u_1\) and common difference \(d\) as shown below.

\[S_n = u_1 + u_1 + d + ... + (u_1 + (n - 2)d) + (u_1 + (n - 1)d)\]

\[2S_n = [u_1 + (u_1 + (n - 1)d)] + [u_1 + (u_1 + (n - 1)d)] + ... + [u_1 + (u_1 + (n - 1)d)]\]

\[S_n = \frac{n}{2}[2u_1 + (n - 1)d]\]

This can be rewritten as follows:

\[S_n = \frac{n}{2}[2u_1 + (n - 1)d]\]

\[= \frac{n}{2}[u_1 + u_n]\]

The sum of a finite arithmetic series is given by

\[S_n = \frac{n}{2}[2u_1 + (n - 1)d] = \frac{n}{2}[u_1 + u_n]\] where \(n\) is the number of terms in the series, \(u_1\) is the first term, \(d\) is the common difference and \(u_n\) is the last term.

International-mindedness

Karl Friedrich Gauss (1777–1855) was a renowned German mathematician. It is said that when he was in primary school his teacher challenged him to find the sum of the numbers from 1 to 100. To the teacher’s amazement, Gauss gave the correct answer almost immediately. He came to the answer by using the method used in investigation 4.
Example 10
The first term of an arithmetic series is 5 and the last term is \(-51\). The series has 15 terms.

a) the common difference
b) the sum of the series.

\[ \begin{align*}
   a &= -51 = 5 + 14d \\
   d &= -36 \\
   S_{15} &= 15 \left( \frac{5 + (-51)}{2} \right) = -345
\end{align*} \]

\[ S_{15} = 15 \left( \frac{5 + (-51)}{2} \right) = -345 \]

Example 11
The first term of an arithmetic series is \(-7\) and the fourth term is 23. The sum of the series is 689. Find the number of terms in the series.

\[ \begin{align*}
   u_1 &= -7 \\
   u_1 + 3d &= 23 \Rightarrow 3d + 23 + 7 &= 10 \\
   S &= 689 = \frac{n}{2} \left( -14 + (n - 1) \cdot 10 \right) \\
   10n^2 - 24n - 1378 &= 0 \\
   5n^2 - 12n - 689 &= 0 \\
   (5n + 57)(n - 13) &= 0 \\
   n &= 13, \text{ since } n \in \mathbb{Z}^+
\end{align*} \]

Why can \( n \) not be a rational or a negative number?

Example 12
Find the value of \( \sum_{r=1}^{28} 5r - 4 \).

\[ \begin{align*}
   u_1 &= 1 \\
   u_{28} &= 140 - 4 = 136 \\
   S_{28} &= \frac{28}{2} (1 + 136) = 1918
\end{align*} \]

Substitute \( r = 1 \) and \( r = 28 \) to find the first and last terms.

Using the formula \( S_n = \frac{n}{2} \left( u_1 + u_n \right) \).

Example 13
The sum of an arithmetic series is given by \( S_n = \frac{n(2a + (n - 1)d)}{2} \). Find the common difference and the first three terms of the series.

\[ \begin{align*}
   S_n &= u_1 - 1 \\
   S_2 &= u_1 + (u_1 + d) \Rightarrow -2 + d = 2 \\
   d &= 4 \\
   u_1 &= -1 \\
   u_2 &= 3 \\
   u_3 &= 7
\end{align*} \]

Exercise 1B
Find the number of terms in each of these sequences:

1. \( 3, 8, 13, 18, \ldots \)
2. \( 101, 97, 93, 89, \ldots \)
3. \( a, a + 1, a + 5, a + 9, \ldots \)
4. \( -20, -5, 10, 25, \ldots \)
5. \( 5, 10, 12, 6, \ldots \)
6. \( -16, 11, 6, \ldots \)
7. \( -5, 10, 25, \ldots \)

Find the 10th term of each of these arithmetic sequences:

1. \( a, a + 1, a + 2, a + 3, \ldots \) 10th term
2. \( 101, 97, 93, 89, \ldots \) 11th term
3. \( a, a + 1, a + 2, \ldots \) 17th term
4. \( -10, -101, -104, \ldots \) 15th term
5. \( 3, 8, 13, 18, \ldots \) 15th term

Find the value of each of the following sums:

1. \( \sum_{r=1}^{28} 5r - 4 \)
2. \( \sum_{r=1}^{54} 7 - 8r \)
3. \( \sum_{r=1}^{10} 2ar - 1, \) where \( a \) is a constant
11 Find the sums of the following sequences up to the term indicated:
   a. 4, –1, –6, … 15th term
   b. 3, 11, 19, … 10th term
   c. 1, –4, –9, … 20th term
12 Calculate the sum of an arithmetic series with 25 terms given that the fifth term is 19 and 10th term is 39.

13 The third term of an arithmetic sequence is –8, and the sum of the first 10 terms of the sequence is –230. Find:
   a. the first term of the sequence
   b. the sum of the first 13 terms.
14 The sum of an arithmetic series is given by \( S_n = \frac{n}{2} [2a + (n-1)d] \). Find the common difference and the first four terms of the series.
15 Calculate the sum of all the odd numbers less than 300.

**Geometric sequences and series**

In Investigation 5, you should have noticed that when filling out the table you would need to multiply the numbers in each row by a particular constant to obtain the following column. In other words, the ratio of a particular term to the previous term is a constant. Such sequences are known as geometric sequences.

If the ratio of two consecutive terms in a sequence is constant then it is a geometric sequence or a geometric progression. We call the constant ratio the common ratio and denote it by \( r \).

Consider how a geometric sequence with first term \( u_1 \) and common ratio \( r \) grows:
- First term: \( u_1 \)
- Second term: \( u_2 = u_1r \)
- Third term: \( u_3 = u_1r^2 \)
- Fourth term: \( u_4 = u_1r^3 \)

This leads to the general term \( u_n = u_1r^{n-1} \).

A geometric sequence with first term \( u_1 \) and common ratio \( r \) has the general term \( u_n = u_1r^{n-1} \), \( r \neq 1 \), \( 0, -1, u_1 \neq 0 \).

**Curiosities in geometric patterns**

- What happens if you have a sequence with first term \( u_1 \) and common ratio 1?  
  - What if the common ratio is 0?  
  - And what happens if the common ratio is –1?  

In the first case, the sequence is just made up of constant terms and is called a uniform sequence.

The next case is a sequence with first term \( u_1 \) and all the other terms are 0, which is an uninteresting sequence.

The third case leads to what is known as an oscillating sequence: \( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots \)

This oscillating sequence becomes particularly interesting if \( u_1 = 1 \), which then leads to the sequence 1, –1, 1, –1, 1, …

If we try to take the sum of this series we run into some curious and interesting results.

We want to look at the sum \( S = 1 + 1 - 1 + 1 - 1 + 1 + \ldots \)
There are various ways of looking at this sum. Possibly the most intuitive way of finding this sum is by grouping the terms into pairs as follows:

\[ S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + 0 + \ldots = 0 \]

But what happens if we pair the terms starting from the second term instead of the first?

\[ S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 1 + 0 + 0 + 0 + \ldots = 1 \]

Yet another result is obtained if we look at the series from a different perspective:

\[ S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \ldots \]

\[ S = \frac{1}{1 - (1/2)} = 1 \]

Why does this paradox arise and which is the correct answer? We have once more stumbled on the concept of infinity. If the number of terms were to be made finite, then the result would be 0 if there are an even number of terms, and 1 if the number of terms were odd, but an infinite sum never ends.

The next examples show how to use the general term formula for a geometric sequence to answer different types of questions.

**Example 14**

Find the common ratio and write the next two terms of each sequence:

- **a** 2.5, 5, 10, …
- **b** 9, 3, 1, …
- **c** \(x, 2x, 4x, \ldots\)

**Solution**

- **a** \(r = \frac{5}{2.5} = 2\)
  
  The next two terms are 20, 40.

- **b** \(r = \frac{3}{9} = \frac{1}{3}\)
  
  The next two terms are \(\frac{1}{3}, \frac{1}{9}\).

- **c** \(r = \frac{2x^3}{x} = 2x^2\)
  
  The next two terms are \(8x^2, 16x^2\).

**International-mindedness**

The series \(S = 1 - 1 + 1 - 1 - 1 + \ldots\) is known as Grandi’s series, after the Italian mathematician Guido Grandi (1671–1742). You may want to look into the history and research on this sum by various mathematicians after its first appearance in Grandi’s book published in 1703.

**Example 15**

Find the number of terms in each of these geometric sequences:

- **a** 0.15, 0.45, 1.35, …, 12.15
- **b** 440, 110, 27.5, …, 0.4296875

**Solution**

- **a**\( u_n = 0.15, r = \frac{0.45}{0.15} = 3\)
  
  \(u_1 = 0.15 \times 3^{n-1} = 12.15\)
  
  \(\Rightarrow 3^{n-1} = 81\)
  
  \(\Rightarrow n = 5\)

  This sequence has 5 terms.

- **b**\( u_n = 440, r = \frac{110}{440} = 0.25\)
  
  \(u_1 = 440 \times 0.25^{n-1} = 0.4296875\)
  
  \(\Rightarrow n - 1 = 5\)
  
  \(\Rightarrow n = 6\)

  This sequence has 6 terms.

**Example 16**

The first term of a geometric sequence is 4 and the common ratio is –2. Determine which term has the value of –2048?

**Solution**

\(4 \times (-2)^{n-1} = -2048\)

\(\Rightarrow n = 10\)

**International-mindedness**

This time the formula uses \((x - 1)\) in the exponent so the answer is \(n = 10\).
The sum of a geometric sequence

When trying to find the value of the series $S = 1 + 3 + 9 + 27 + 81 + 243$, Max notices that this is a geometric series with common ratio $3$, and that if he were to multiply the series by $3$, he could more easily calculate the sum as follows:

$$3S = 3 + 9 + 27 + 81 + 243 + 729$$

Thus:

$$S = \frac{3S - S}{2} = \frac{729 - 1}{2} = 364$$

Example 18

The first term of a geometric sequence is $16$ and the common ratio is $\frac{1}{2}$. Find the biggest term that is smaller than $1000$.

$$16 \times \left(\frac{1}{2}\right)^{n-1} < 0.001$$

$\Rightarrow n = 15$

Alternatively, you can use your GDC.

Investigation 6

In the diagram, $AB$ represents a piece of string which is $100$ cm long.

The string is cut in half and one of the halves, $CD$, is placed underneath. The remaining half is now cut in half and one of the halves, $DE$, is placed next to $CD$. The process is continued as shown in the diagram.

1. Copy and complete the table below.

<table>
<thead>
<tr>
<th>Line segment</th>
<th>Length of string segment (cm)</th>
<th>Total length of segments (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>DE</td>
<td>25</td>
<td>75</td>
</tr>
<tr>
<td>EF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FG</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. **Factual!** As this process continues indefinitely, what do you notice about the length of the line segments? What about the total length of segments?

Modelling this scenario mathematically:

$$CD = 50 \text{ cm}$$

$$DE = 50 \times \frac{1}{2} = 25 \text{ cm}$$

$$EF = DE \times \frac{1}{2} = 25 \times \frac{1}{2} = 12.5 \text{ cm}$$

$$CD + DE + EF + FG = 50 + 25 + 12.5 + 6.25 = 83.75 \text{ cm}$$

Max then tried to generalize this result for a finite geometric series with common ratio $r$ and having $n$ terms as follows:

$$S_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1}$$

$$rS_n = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^n$$

$$\Rightarrow S_n = \frac{u_1 (1 - r^n)}{1 - r}, \quad r \neq 1$$

The sum of a finite geometric series is given by

$$S_n = \frac{u_1 (1 - r^n)}{1 - r}, \quad r \neq 1$$

where $n$ is the number of terms, $u_1$ is the first term and $r$ is the common ratio.

HINT

This formula can also be written as follows:

$$S_n = \frac{u_1 (1 - r^n)}{1 - r}, \quad r \neq 1$$

This makes calculations easier when $r > 1$.

The sum of a geometric sequence

When trying to find the value of the series $S = 1 + 3 + 9 + 27 + 81 + 243$, Max notices that this is a geometric series with common ratio $3$, and that if he were to multiply the series by $3$, he could more easily calculate the sum as follows:

$$3S = 3 + 9 + 27 + 81 + 243 + 729$$

Thus:

$$S = \frac{3S - S}{2} = \frac{729 - 1}{2} = 364$$

Example 17

The fourth term of a geometric sequence is $54$ and the sixth term is $486$. Determine the possible values of the common ratio.

$$u_4 = u_1 \times r^3 = 54$$

$$u_6 = u_1 \times r^5 = 486$$

$$\Rightarrow r^3 = \frac{486}{54} = 9$$

$$\Rightarrow r = \pm 3$$

Or

$$u_4 = u_1 \times r^3$$

$$\Rightarrow r^3 = \frac{486}{54} = 9$$

$$\Rightarrow r = \pm 3$$

Use $u_n = u_1 \times r^{n-1}$.

Divide the two expressions to obtain $r^2$.

$$\Rightarrow r^2 = \frac{486}{54} = 9$$

$$\Rightarrow r = \pm 3$$

$$u_n = u_1 \times r^{n-1}.$$
After four cuts have been made the sum of the length of string segments placed next to each other is a geometric sequence with four terms. Show that if \( n \) cuts are made this sum becomes

\[
50 \times \left(1 - (0.5)^n\right) \quad \frac{1}{1 - 0.5}
\]

Enter this into a table as shown below to see what happens as \( n \) gets bigger.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 50 \times \left(1 - (0.5)^n\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
</tr>
<tr>
<td>3</td>
<td>87.5</td>
</tr>
<tr>
<td>4</td>
<td>93.75</td>
</tr>
<tr>
<td>5</td>
<td>98.4375</td>
</tr>
<tr>
<td>6</td>
<td>99.21875</td>
</tr>
<tr>
<td>7</td>
<td>99.609375</td>
</tr>
<tr>
<td>8</td>
<td>99.90234375</td>
</tr>
</tbody>
</table>

3 What would happen if you repeated this experiment, but this time you cut \( CD \) to be \( \frac{2}{3} \) of \( AB \) and \( DE \) to be \( \frac{3}{4} \) of the remaining piece of string?

4 Repeat the process using \( CD \) to be \( \frac{1}{2} \) of \( AB \) and \( DE \) to be \( \frac{1}{4} \) of the remaining piece of string. What if the fraction used was \( \frac{2}{3} \)?

5 Write a short reflection on your results which includes answers to the following questions:

- **Factual** Why were you asked to change the length of the string cut?
- **Conceptual** How has this process helped you analyse the situation?
- **Conceptual** What conclusion could you draw from your analysis?

### Convergent and divergent series

An infinite geometric series is **convergent** when the sum tends to a finite value as the number of terms gets bigger. If a geometric series does not converge it is said to be **divergent**.

In Investigation 6, the series always converged to 100 cm, the length of the original piece of string.

### Investigation 7

In Investigation 6, you would have noticed that you had a geometric series in each case. You will now investigate a general geometric series in order to understand which conditions will make the series converge.

For a geometric series, we know that

\[
S_n = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1
\]

The sum of \( n \) terms of a geometric series is

\[
S_n = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1
\]

When \(-1 < r < 1\), \( r^n \) approaches zero for very large values of \( n \). The series therefore converges to a finite sum given by

\[
S = \frac{u_1}{1 - r}
\]

The next examples demonstrate how to use the formulae for sums of finite and infinite geometric series.

### Example 19

A geometric series has first term 3 and common ratio 2. Find the sum of the first five terms.

\[
S_5 = \frac{3(1 - 2^5)}{1 - 2} = \frac{3(1 - 32)}{-1} = 93
\]

Use the formula

\[
S = \frac{u_1(1 - r^n)}{1 - r}
\]
Example 20
Calculate the geometric series given by $\sum_{i=1}^{n} 2 \times \left(\frac{1}{2}\right)^i$.

$u_i = 1, r = \frac{1}{2}$

$S_n = \frac{\left(\frac{1}{2}\right)^1 - 1}{1 - \frac{1}{2}} = \frac{\frac{1}{2} - 1}{\frac{1}{2}} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$  

$S = \frac{\left(\frac{1}{2}\right)^1 - 1}{1 - \frac{1}{2}} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$

Or using a GDC: $\sum_{i=1}^{n} 2 \times \left(\frac{1}{2}\right)^i = 1.984375$

Example 21
Find two possible geometric sequences where the sum of the first two terms is 20 and the sum of the first four terms is 1640, and write the general term of each sequence.

$S_2 = u_1 + u_2 = u_1(1 + r)$

Find the ratio of $S_2$ to $S_1$.

$r^2 = 81$  

$r = 9$ or $-9$

Calculate $u_i$ for each value of $r$.

$u_2 = 2 \times 9^{-1}$ or $u_2 = \left(\frac{5}{2}\right) \times 9^{-1}$

Example 22
The sum of the first $n$ terms of a geometric sequence is given by $S_n = 7^n - 1$. Find the first term and the common ratio of the sequence.

$S_1 = 7 - 1 = 6 \Rightarrow u_1 = 6$

$S_2 = 49 - 1 = 48 \Rightarrow 6 + u_2 = 48 \Rightarrow u_2 = 42$

$r = \frac{42}{6} = 7$

Example 23
Determine how many terms are required for the sum of the geometric series given by $\sum_{i=1}^{n} \frac{1}{2} \times 2^i$ to exceed 1000.

$S_1 = 6, S_2 = 2^2 - 1 = 3$

Use technology to produce a table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
</tr>
</tbody>
</table>

Example 24
For what values of $x$ does the series $\sum_{i=1}^{8} \left(1 + \frac{x}{2}\right)^i$ converge? Find the sum when $x = -1.5$.

$S = \left(1 + \frac{x}{2}\right)^1 + \left(1 + \frac{x}{2}\right)^2 + \left(1 + \frac{x}{2}\right)^3 + ...$

Write the first three terms of the series.

This is a geometric series where

$r = \left(1 + \frac{x}{2}\right)$

Identify $r$. 

Continued on next page
For convergence:

\[-1 < \left(1 + \frac{x}{2}\right) < 1\]

\[\Rightarrow -2 < \frac{x}{2} < 0\]

\[\Rightarrow -4 < x < 0, \text{ but } x \text{ cannot be } -2\]

When \(x = -1.5\), the series converges to

\[S = -0.5\]

\[\frac{1}{1.5}\]

Use condition for convergence, i.e., \(-1 < r < 1\).

Example 25

A geometric series converges to 8. The second term of the series is \(-\frac{3}{2}\). Find the common ratio.

\[S = \frac{u_1}{1-r} = 8\]

\[u_2 = u_1r = -\frac{3}{2}\]

\[16r^2 - 16r - 5 = 0\]

\[\Rightarrow (4r + 1)(4r - 5) = 0\]

\[r = \frac{1}{4} \text{ or } r = \frac{5}{4}\]

But, since the series converges, \(-1 < r < 1\)

Hence the answer is \(r = \frac{1}{4}\)

Exercise 1C

1. Write down the fifth term and the general term of each of the following sequences:
   a. 1, 3, 9, ...
   b. 8, 4, 2, ...
   c. \(x, x^2, x^3, \ldots\)
   d. -3, 3, -3, 3, ...

2. Determine the common ratio and write the terms indicated in each of the following sequences:
   a. 63, 21, 7, ...
   b. 243, 81, 27, ...
   c. \(\frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \ldots\)

3. Determine the number of terms in each of the following sequences:
   a. 0.02, 0.06, 0.18, ..., 3936.6
   b. 64, 32, 16, ..., \(\frac{1}{128}\)

4. The fourth term of a geometric sequence is 6 and the seventh term is 48. Find the common ratio and the first term of the sequence.

5. The third term of a geometric sequence is 6 and the fifth term is 54. Find the two possible values of the common ratio and the sixth term of each sequence.

6. The first term of a geometric sequence is 9 and the fifth term is 16. Show that there are two possible sequences and find their common seventh term.

7. The numbers \(3a + 1, a + 2, a - 4\) are three consecutive terms of a geometric sequence. Find the two possible values of the common ratio.

8. The numbers \(a - 1, a + 1, a - 2\) are the fourth, fifth and sixth terms, respectively, of a geometric sequence. Find the common ratio and the first term of the sequence.

9. Find the following series for the number of terms stated:
   a. \(3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots\)
   b. \(8 + 4 + 2 + 1 + \ldots\)
   c. \(0.1 + 0.03 + 0.009 + 0.0027 + \ldots\)
   d. \(0.1 - 0.03 + 0.009 - 0.0027 + \ldots\)

10. Calculate:
   a. \(\sum_{i=1}^{15} x^i\)
   b. \(\sum_{i=1}^{15} 5x^i\)

11. Show that a geometric sequence with first term 3 and seventh term 2 has two possible sums to infinity and find them.

12. The sum of \(n\) terms of a certain series is given by \(S_n = \frac{1}{2^n} - 1\).
   a. Find the first three terms of the series.
   b. Show that the terms of the series are in geometric progression.

13. The second term of a geometric series is 28 and the third term is 28(1 - \(a\)). Find the common ratio, given that the series converges and the sum of the first three terms is 147.

14. A length of material measures 2 m. It is cut into three lengths which are in geometric progression. The longest piece is twice as long as the shortest piece. Find the common ratio of the sequence and the exact length of the shortest piece.

15. Write the first four terms of the series \(\sum_{i=1}^{15} (-1)^{i+1} \left(\frac{x}{2} + 1\right)^i\). Determine for what values of \(x\) this series converges. Find the value of the series when \(x = -0.8\).

Developing inquiry skills

Go back to the original question about Koch’s snowflake and try to address the following, assuming that the length of each side of the original triangle is 81 cm:

- Calculate the perimeter of the snowflake at each iteration.
- Calculate the area of the snowflake at each iteration.
- Tabulate the results and explain the number patterns that you observe.
- Create a model that helps you generalize the perimeter and area at any iteration.

Although it might not be obvious, you have actually been exposed to arithmetic and geometric sequences and series in previous mathematics classes. Usually this was in the form of solving word problems. When we apply knowledge that was obtained by abstracting generalizations of mathematical concepts to real-life situations we
are actually modelling the situation mathematically. The following investigations illustrate how arithmetic sequences can be hidden in everyday practices.

**Investigation 8**

Before the start of the school year, a stationer needs to stock up with notebooks. He has been given the following offers:

Provider A: Notebooks in packs of 20, at an offer of 6 for the price of 4, where each packet of 20 notebooks costs €10.

Provider B: Notebooks in packs of 100, at an offer of 3 for the price of 2, where each pack of 100 costs €48.

The stationer is considering stocking between 500 and 3000 notebooks, in multiples of 100.

1. **Factual** The stationer first looks at the offer made by Provider A and realizes that he would get the cheaper rate when the number of notebooks ordered are in a particular arithmetic sequence. Show that the stationer is correct.

2. **Factual** The stationer’s wife tells him that the argument is true for the offer from Provider B. Is the wife also correct?

3. **Conceptual** They then compare costs incurred when buying notebooks from Provider A and from Provider B. They notice that for certain numbers of notebooks ordered it would be cheaper or the same rate if they were to order from Provider A. Determine which numbers they are referring to. How is this list of numbers different from the previous two answers?

4. **Conceptual** The stationer would like to divide his order of 1500 notebooks between the two providers. How can he divide up his order to minimize his cost price? If he sells the notebooks at 45 cents each, what would be his percentage profit? How does this compare to his percentage profit had he ordered all the notebooks from either of the providers?

**Investigation 9**

Students were given the following data and asked to work in groups to create a growth model for the shipment of smartphones worldwide from 2011 to 2016.

<table>
<thead>
<tr>
<th>Year</th>
<th>Shipments worldwide in billions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2011</td>
<td>0.52</td>
</tr>
<tr>
<td>2012</td>
<td>0.74</td>
</tr>
<tr>
<td>2013</td>
<td>1.05</td>
</tr>
</tbody>
</table>

Students in group A decided that the data could be modelled using an arithmetic sequence by taking the average difference.
1. When Jacob turned 18 he had access to the money his grandparents had invested in a savings account. He decided to reinvest $10,000 at a compound interest rate of 3% each year. He decided to add $200 to this investment on his next birthday and each following birthday until he turned 25. Evaluate how much money was in his account just after his 20th birthday: 

\[ \text{Amount after 1 year} = 10,000 \times 1.03 + 200 \] 

2. Just before his 19th birthday he had 

\[ 10,000(1.03) \] 

3. Just before his 19th birthday he had 

\[ 10,000(1.03)^2 + 200 \] 

4. After 7 years, just after his 25th birthday the amount would be: 

\[ 10,000(1.03)^7 + 200(1.03)^4 \] 

5. Geometric series with \( u_1 = 1 \), \( r = 1.03 \) and \( n = 7 \). 

\[ \text{Total interest} = 13,831 \times 140 = $2,431 \]

Exercise 1D

1. In 2010, a shop sold 230 televisions. Every six months the shop sold five more televisions, so that it sold 230 televisions in 2011, 240 in 2012, etc.

   a. Evaluate how many televisions the shop sold in 2017.

   b. Calculate the total number of televisions sold between 2010 and 2017 inclusive.

   c. The selling price of a television was $600 in 2010 but the selling price fell by $20 each year. In a particular year the number of televisions sold by the shop was half of the selling price of each television. Determine in which year this occurred.

2. Jane started working in 2000. On each successive year she received a salary increase equivalent to 1.5% of her previous salary. In 2011 her salary was €49,650. Determine her starting salary to the nearest €.

3. Carla traced her family tree back four generations. Carla’s parents are the first generation back and her first set of ancestors. Carla’s four grandparents are the second generation back and her second set of ancestors. Determine in which year this occurred.

4. Prisana is testing a recipe for a cake and tries it out several times, adjusting the amount of flour and sugar used each time. In the first recipe she uses 200 g of flour, and she decides to increase the weight of flour by 20 g in each trial. She has time to try out the recipe just 10 times. How many kg of flour would she need?
a If the sides of the largest square have a length of 2 units, calculate the side lengths of the second, third and fourth squares. A spiral is formed by joining segments shown as red lines in the second diagram.
b Use your answers in part a to find the length of the spiral shown.
c Explain what happens to the length of the spiral if we continue the process infinitely.
A different spiral is formed by shading the triangles as shown in the third diagram.
d Find the total area of the shaded triangles.
e Determine the total area of the spiral formed if the process of forming squares and shading triangles is continued infinitely.

6 Gyuhun takes out a loan of $1500 to furnish his new student apartment. The terms of the loan are that Gyuhun will pay equal monthly instalments. Interest is calculated monthly and is charged at 12% p.a. The loan is to be repaid in two years.
a Calculate the amount that Gyuhun has to pay each month if the first repayment is made one month after the money is borrowed and after interest is calculated.
b Evaluate how much, to the nearest dollar, Gyuhun actually has to pay for furnishing the apartment?

7 An architect is designing a cinema that should hold 570 seats. The front row of the cinema can hold 30 seats and each consecutive row is to hold six seats more than the previous row.
a Evaluate how many rows the cinema can hold.
For optimal viewing, the floor of the cinema needs to be stepped so that the step between each row is 15 cm high and 95 cm deep. Assuming that the front row is 3 m away from the screen and the ceiling at the back row is 2.4 m high, determine:
b the horizontal distance from the screen to the top row
c the maximum height of the ceiling.

8 Ayla, Brynna and Cindy each receive $200 from their parents with the condition that they promise to invest it for at least 10 years. Ayla invests her money in an account with Rapid Bank that offers 5% simple interest annually. Brynna says it is better to invest in an account that offers compound interest because it grows quicker, so she uses an account with Quick Bank that offers 3.5% interest compounded annually. Cindy is not sure which offer is best so she invests $100 with Rapid Bank and $100 with Quick Bank. Answer the following questions giving your answers to the nearest €.
a Evaluate how much each investment is worth after 10 years.
b How much is each investment worth after 25 years?
c Determine after how many years the three investments yield approximately the same amount.

9 At the start of 2010 Karim had $5000 to invest. He decided to invest part of the money in a savings account that offered 1.3% simple interest per year. He added $1000 to this amount and fixed it for 10 years in bonds that offered 2.5% compound interest per year. The rest of the money he invested in shares.
a After one year the money invested in shares made a loss of 1%. Given that the total amount of money invested increased by $75, determine how much money Karim invested in each.
b At the end of the first year, Karim decided to sell the shares at their current value and reinvest the money in the savings account. Evaluate the total value of his investment at the end of 2020.

c Evaluate how much more money he would have made if he had divided up the $5000 equally between the savings account and bonds at the very start? (Give all your answers to the nearest dollar.)

10 A pharmaceutical company has developed a drug that fights a bacterial infection. The drug is to be administered four times per day, every six hours. It was found that six hours after administering, 37.5% of the original amount was still in the bloodstream. The maximum safe level of the drug in the bloodstream is 8 mg ml
1.
Evaluate how much more drug the company advises that the drug should not be administered for more than 10 days. Evaluate the maximum amount that should be administered to ensure that the amount of drug in the bloodstream does not exceed the safety level?

The drug starts being effective when the amount of drug in the bloodstream is 7 mg ml
1. Determine how many times the drug must be administered for this level to be reached.

### Developing inquiry skills
You should now go back to the start of the chapter and answer the questions in the scenario that involves taking a loan for buying a new car.

### Developing your toolkit
Now do the Modelling and investigation activity on page XX.

## 1.3 Proof

Each of the three diagrams below, not drawn to scale, consists of a square ABCD of different sizes. Line PR is perpendicular to AB and DC and line SQ is perpendicular to AD and BC. Copy and complete the table on the next page.
1. Describe the relationship between the areas of square ABCD, rectangles PBCT and TRDS, and square TQCR.
2. Rewrite the relationship above replacing words with numbers.
3. Now rewrite the relationship for the diagram on the right.
4. Factual What do you call this relationship? Why?

Area ABCD Area APTS Area BPQT Area STRD Area TQCR

\[(3 + 4)^2 = 49\] \[4^2\] \[3 \times 4\]

Investigation 10b
Each of the three diagrams below, not drawn to scale, consists of two squares ABCD and PQRS. Use this information to copy and complete the table for each diagram.

<table>
<thead>
<tr>
<th>Area ABCD</th>
<th>Area PQRS</th>
<th>Area (\triangle PBQ)</th>
<th>Area PQRS + 4 \times Area (\triangle PBQ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3 + 4)^2 = 49)</td>
<td>(x^2)</td>
<td>6</td>
<td>(x^2)</td>
</tr>
</tbody>
</table>

1. Describe the relationship between the areas of the squares and the triangles.
2. Rewrite the relationship above replacing words with numbers and making the area of PQRS the subject.

What is proof?
In Investigation 10a, you should have noticed that by making the question visual, you were able to prove the identity \((a + b)^2 = a^2 + 2ab + b^2\). By looking at a square which is divided into two smaller squares and two rectangles and comparing areas you came to a valid conclusion, and by then representing numbers with variables you could show that this identity is valid for all values of \(a\) and \(b\). We can say that you have proved the statement \((a + b)^2 = a^2 + 2ab + b^2\) is true for all \(a, b \in \mathbb{R}\) because the perpendiculars PR and SQ could be placed anywhere along the sides of square ABCD, which could also be as large or as small as you wanted.

Similarly, in Investigation 10b, you were able to validate the statement that in any right-angled triangle the lengths of the sides obey the relationship \(a^2 + b^2 = c^2\), where \(c\) is the length of the hypotenuse of the triangle.

A proof in mathematics often consists of a logical set of steps that validates the truth of a general statement beyond any doubt.

There are many ways of presenting a proof and we will be looking at some of these ways in this section.

Types of proof

Investigation 11
Copy and complete the table below and then suggest a conjecture.

\[
\begin{align*}
1 & \quad 1 + 3 \\
1 + 3 & \quad 1 + 3 + 5 \\
1 + 3 + 5 & \quad 1 + 3 + 5 + 7 \\
\end{align*}
\]
Example 28
Show that \(1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2\).

Write out the sum in reverse order.

Add the two sums and simplify.

```
\begin{align*}
S &= 1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) \\
\Rightarrow S &= (2n - 1) + (2n - 3) + \ldots + 1 \\
&= 2(n - 1) + 2(n - 3) + \ldots + 2 + 3 + 1 \\
&= 2S - n \\
\Rightarrow S &= \frac{n(n + 1)}{2}
\end{align*}
```

A direct proof is a way of showing the truth of a given statement by constructing a series of reasoned connected established facts. In a direct proof the following steps are used:

- Identify the given statement.
- Use axioms, theorems, etc, to make deductions that prove the conclusion of your statement to be true.

The proof given in Example 28 consists of a set of reasoned steps that leads to the required result. Note that we did not just quote the sum of an arithmetic sequence with first term 1 and common difference 2, although by the definition of direct proof this would have been valid.

Example 29
Show that:

**a** The sum of an odd and even positive integer is always odd.

**b** The sum of two even numbers is always even.

**c** The sum of two odd numbers is always even.

- **a** Let \(m\) and \(n\) be an odd and an even positive integer respectively.
  \(\Rightarrow m = 2p - 1\) and \(n = 2s\) where \(p, s \in \mathbb{Z}^+\)
  \(\Rightarrow m + n = 2p - 1 + 2s = 2(p + s) - 1\)
  This is an odd number since \(p + s \in \mathbb{Z}^+\).

- **b** Let \(m\) and \(n\) be two even positive integers.
  \(\Rightarrow m = 2p\) and \(n = 2r\) where \(p, r \in \mathbb{Z}^+\)
  \(\Rightarrow m + n = 2(p + r)\)
  This is an even number since \(p + r \in \mathbb{Z}^+\).

- **c** An odd positive integer can be written as \(2k - 1\) where \(k \in \mathbb{Z}^+\).
  An even positive integer can be written as \(2k\) where \(k \in \mathbb{Z}^+\).
Let $m$ and $n$ be two odd positive integers.

\[ \Rightarrow m = 2p - 1 \text{ and } n = 2s - 1 \text{ where} \\
\Rightarrow p, s \in \mathbb{Z}^* \\
\Rightarrow m + n = 2p + 2s - 2 \\
\Rightarrow m + n = 2(p + s - 1) \\
\text{This is an even number since} \\
p + s - 1 \in \mathbb{Z}^*. \\

**Example 30**

Show that \( (x + a)^2 - (a^2/4) = x^2 + ax \).

\[
\text{LHS} = (x^2 + \frac{a}{2}x + \frac{a^2}{4}) - \frac{a^2}{4} \\
= x^2 + ax
\]
Expand and simplify.

In Example 30 we started from the left hand side and showed that this is equivalent to the right hand side. **When writing down a proof it is very important to work on one side of the statement only.** A proof is also valid if you work on each side consecutively to obtain the same result. This method is shown in the example below.

**Example 31**

Prove that \((n + 4)^2 - 3n - 4 = (n + 1)(n + 4) + 8\).

\[
\begin{align*}
\text{LHS} &= (n^2 + 8n + 16) - 3n - 4 \\
&= n^2 + 5n + 12 \\
\text{RHS} &= n^2 + 4n + 4 + 8 \\
&= n^2 + 5n + 12 \\
\text{LHS} &= \text{RHS} \\
(n + 4)^2 - 3n - 4 &= (n + 1)(n + 4) + 8
\end{align*}
\]

Note that you work on each side separately and not in the same line.

**Example 32**

Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

Let \( n \) be a four-digit number such that
\[
n = d_1d_2d_3d_4.
\]
We know that:
\[
\begin{align*}
d_1 &= p \times 10^3 \\
d_2 &= q \times 10^2 \\
d_3 &= r \\
d_4 &= s
\end{align*}
\]
where \(0 \leq p, q, r, s \leq 9\) and \( p \neq 0 \).

Therefore
\[
\begin{align*}
n &= p \times 10^3 + q \times 10^2 + r \times 10^1 + s \times 10^0 \\
&= p(10^3) + q(10^2) + r(10^1) + s(10^0) \\
&= 999p + 99q + 9r + 3s
\end{align*}
\]

Since we are given that \( n \) is a four-digit number, \( p \neq 0 \).

We are given that the sum of the digits is a multiple of 3.

**Example 33**

Show that \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1}{2} \).

\[
\begin{align*}
\text{LHS} &= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right) \\
&= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right) \\
&= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right) + \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \\
&= \frac{1}{2} \quad \text{(Converging geometric series)} \\
&= \frac{1}{2} \quad \text{(Converging geometric series)} \\
&= \frac{1}{2} \quad \text{(Converging geometric series)}
\end{align*}
\]

Separate the given series into one series with positive terms and one with negative terms.

Note that this is the difference of two converging geometric series.

Use the formula for sum of converging geometric series and simplify.
Exercise 1E

1. Prove that \((a + b)^2 + (a - b)^2 = 2(a^2 + b^2)\).
2. Show that the product of two odd numbers is always an odd number.
3. Prove that a four-digit number is divisible by 9 if the sum of its digits is divisible by 9. Hence identify which of the numbers 3978, 5453, 7898, 9864, 5670 are divisible by 9 without carrying out any division.
4. Show that \((a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2\).
5. Prove that \(\frac{1}{3} \cdot \frac{2}{9} \cdot \frac{2}{27} \cdot \frac{2}{81} \cdot \frac{2}{243} \cdot \frac{2}{729} = \frac{1}{8}\).
6. Prove that the difference between the squares of two consecutive numbers is an odd number.
7. Show that \(\frac{1}{n(n + 1)} + \frac{1}{n(n + 2)} + \frac{1}{n(n + 3)} = \frac{n^2 + 1}{n(n^2 - 1)}\). Hence determine the value of \(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\).

Proof by contradiction

Investigation 12

You will now look at the statement below and answer the questions. If \(n \in \mathbb{Z}\) and \(5n + 2\) is even, then \(n\) is even.

1. Write a direct proof of this statement.
2. Now assume that \(n\) is an odd number and rewrite the expression \(5n + 2\) to reflect this.
3. Simplify your expression and explain why this can never be an even number.
4. **Factual** How is the second method different to a direct proof?

In Investigation 12 you managed to find a different argument to prove the statement by using the contrapositive. You started by assuming that the second part of the statement is false and showed that this led to a contradiction, i.e., the first part of the statement was also false. In this case, the statement could easily be proved directly but this is not always the case, and sometimes you will need to use the second method to prove a statement correct.

Investigation 13

In geometry, the triangle inequality states that in any triangle ABC, the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side.

Applying this to the triangle on the right we get the following:

\[ a + b \geq c \]
\[ a + c \geq b \]
\[ b + c \geq a \]

1. Explain the situation when:
   \[ a + b = c \]
   \[ a + c = b \]
   \[ b + c = a \]
2. Look at triangle ABC in the quadrilateral on the right and write the triangle inequality in terms of AB and DC.
3. Now apply the triangle inequality to triangle ABC in terms of AC and AB.
4. Look at the inequalities obtained in questions 2 and 3 and comment on your result.
5. Now draw diagonal BD and repeat steps 3 and 4 for triangles ABD and DBC.
6. What happens if you change the order of the sides?
7. **Factual** What do you conclude from this investigation?
8. **Conceptual** How else could you have come to the same conclusion?

When setting out a proof by contradiction you follow the following steps:

- Identify what is being implied by the statement.
- Assume that the implication is false.
- Use axioms, theorems, etc … to arrive at a contradiction.
- This proves that the original statement must be true.

In other words, the assumption contradicts either a given statement or something we already know to be true, or in some cases both.

**TOK**

What do mathematicians mean by mathematical proof, and how does it differ from good reasons in other areas of knowledge?

**International-mindedness**

Reductio ad absurdum (reduce to absurdity) is a term used to describe logical reasoning that attempts to disprove a statement by showing that it leads to an absurd result. In fact, this method can be traced back to the Greek philosopher Aristotle where he talks about "reduction to the impossible" in his book *Prior Analytics*.
The following examples will help you understand this method of proof so that you can then apply it in set tasks.

**Example 34**
Prove by contradiction:

- if the integer \( n \) is odd then \( n^2 \) is also odd
- if \( n^2 \) is even then \( n \) is also even

- **a** Assume that \( n^2 \) is even.
  \[ n^2 = 2k, \ k \in \mathbb{Z} \]
  \[ n \times n = 2k \]
  But this cannot be true if \( n \) is odd as we know that the product of two odd numbers is also odd.
  Hence given that \( n \) is odd \( n^2 \) is also odd.

- **b** Assume that \( n \) is an odd integer.
  \[ n = 2k \pm 1, \ k \in \mathbb{Z} \]
  \[ n^2 = (2k \pm 1)^2 = 4k^2 \pm 4k + 1 \]
  \[ n^2 = 2(2k^2 \pm 2k) + 1 = 2p + 1 \]
  Which is an odd number.
  Hence given that \( n^2 \) is an even integer \( n \) is also an even integer.

- **Start by making an assumption that the resulting statement is false.**
- **We want to prove that \( n^2 \) even \( \Rightarrow \) \( n \) even.**
- **To use the contrapositive we start with \( n \) and deduce the contradiction i.e. that \( n^2 \) is not even.**

**Example 35**
Show that \( \sqrt{2} \) is irrational.

Assume that \( \sqrt{2} = \frac{m}{n} \) where \( m, n \in \mathbb{Z} \) and \( m, n \) have no common factors.

\[ 2 = \frac{m^2}{n^2} \]
\[ m^2 = 2n^2 \]
This means that \( m \) is an even integer.

\[ m = 2k \]
\[ m^2 = 4k^2 \]
This leads to:

\[ m^2 = 4k^2 = 2n^2 \]
\[ n^2 = 2k^2 \]
This means that \( n \) is an even integer.

**How did the Pythagoreans find out that \( \sqrt{2} \) is irrational?**

- **Make the assumption that the statement is false, i.e. \( \sqrt{2} \) is rational.**
- **Here we are using the fundamental theorem of arithmetic which states that every integer bigger than 1 is either prime or a multiple of primes and since \( n \) is an even integer it must contain a prime factor of 2.**

**Example 36**
Prove that there is no \( x \in \mathbb{R} \) such that \( \frac{1}{x-2} = 1-x \).

Assume there exists a real number \( a \) such that \( \frac{1}{a-2} = 1-a \).

\[ 1 = (a-2)(1-a) \]
\[ 1 = -a^2 + 3a - 2 \]
\[ a^2 - 3a + 3 = 0 \]
\[ a = \frac{3 \pm \sqrt{9-12}}{2} \in \mathbb{R} \]

This cannot be true as 3 is not even.

**Assume that a real solution \( x = a \) exists.**
**Solve for \( a \).**
**Apply the quadratic formula and conclude that \( a \) is not a real number since the number under the square root is negative.**
**Show that \( a \) cannot be a real number.**

**Example 37**
Prove that if \( m, n \in \mathbb{Z} \), then \( m^2 - 4n - 7 \neq 0 \).

Assume that \( m^2 - 4n - 7 = 0 \).

\[ m^2 = 4n + 7 \]
\[ m^2 = 4n + 6 + 1 \]
\[ m^2 = 2(2n + 3) + 1 \]
This means that \( m^2 \) is an odd integer.

But this means that \( m \) is an odd integer since an even integer squared is even.

Let \( m = 2p + 1 \) where \( p \in \mathbb{Z} \).

Then
\[ (2p + 1)^2 - 4n - 7 = 0 \]
\[ 4p^2 + 4p + 1 - 4n - 7 = 0 \]
\[ 2p^2 + 2p - 2n = 3 = 0 \]
\[ 2(p^2 + p - n) = 3 \]
Since \( p, n \in \mathbb{Z} \) this cannot be true as 3 is not even.

Hence \( m^2 - 4n - 7 \neq 0 \) given that \( m, n \in \mathbb{Z} \).
Sometimes we encounter statements that seem to be true, and for every example that we consider the statement seems to hold. Although such examples are good to verify a statement or conjecture, they are not sufficient to prove the statement. It takes only one example that contradicts the statement to justify that the statement is wrong.

A counterexample, or counterclaim, is an acceptable “proof” of the fact that a given statement is false.

Some examples follow to demonstrate how counterexamples can be used.

**Example 38**
Show by a counterexample that the following statements are not true.
- a. If \( n \in \mathbb{Z} \) and \( n^2 \) is divisible by 4, then \( n \) is divisible by 4.
- b. If \( n \in \mathbb{Z} \) then \( n^2 + 1 \) is a prime number.
- c. If an integer is a multiple of 10 and 12 then it is a multiple of 120.

| a. When \( n = 6 \), \( n^2 = 36 \) which is divisible by 4, but 6 is not divisible by 4. |
| b. \( n = 3 \Rightarrow n^2 + 1 = 10 \) which is not a prime number. |
| c. 60 is a multiple of both 10 and 12 but it is not a multiple of 120. |

Again, the statement is true for many integers but we only need one counterexample.

**Exercise 1F**
In questions 1 to 9, prove the statements by contradiction.

1. For all \( n \in \mathbb{Z} \), if \( n^2 \) is odd then \( n \) is also odd.
2. \( \sqrt{3} \) is irrational.
3. \( \sqrt{2} \) is irrational.
4. For all \( p, q \in \mathbb{Z} \), \( p^2 - 8q - 11 \neq 0 \).
5. For all \( a, b \in \mathbb{Z} \), \( 12a - 6b \neq 0 \).
6. If \( a, b, c \in \mathbb{Z} \), where \( c \) is an odd number and \( a^2 + b^2 = c^2 \), then either \( a \) or \( b \) is an even number.
7. If \( n, k \in \mathbb{Z} \), then \( n^2 + 2 \neq 4k \).
8. If \( r \) is an irrational number and \( q \) is a rational number then \( p + q \) is also irrational.

9. Given that \( m \) and \( n \) are positive integers, it follows that \( m^2 - n^2 \neq 1 \).

10. Show by a counterexample that the following statements are not true:
- a. \( (m + n)^2 = m^2 + n^2 \)
- b. If a positive integer is divisible by a prime number then the number is not prime.
- c. \( 2^n - 1 \) is a prime number for all \( n \in \mathbb{N} \).
- d. \( 2^n - 1 \) is a prime number for all \( n \in \mathbb{Z}^+ \).
- e. The sum of three consecutive positive integers is always divisible by 4.
- f. The sum of four consecutive positive integers is always divisible by 4.

**Proof by induction**

**Investigation 14**
Look at the diagrams and answer the questions.

1. Each of the three diagrams represents a series. Write them down.
2. If the diagrams were to continue, what would the next three diagrams be?
3. Write a conjecture based on your findings.
4. Prove your conjecture using a direct proof.

In Investigation 14, you used a visual representation of a series to make a conjecture about a special series and then prove it. In this case, you were able to prove the conjecture directly, but there are times when such a direct proof is not possible. Sometimes we need to revert to a different proof which is called **proof by induction**.

To illustrate the principle of proof by induction, imagine two dominoes placed standing at a distance less than half their length, as illustrated in the diagram on the right. If the first domino is knocked over it will fall and cause the second domino to fall with it. This is the starting point of the process and is called the **basic step**.

Now assume that the domino in the 4th place falls if the domino before it (in the \((k - 1)\)th position) falls. This assumption is the second step in the process.

If we were to add another domino at the end of the \(k\) dominoes, this last domino will also fall. This analogy represents the final step of the process which is called the **inductive step**.

We can then finalize our argument by stating that at the start of the process it was shown that the first domino caused the second domino to fall. We can now use the second and third step that a third domino placed behind the first step will also fall, and again using the two steps, a fourth domino will also topple over. We can continue repeating this process as many times as we want. In other words, we have shown that we can have as many dominoes as we like and they will all fall if the first domino knocks over the second domino.

We apply the dominoes analogy to mathematics to prove the statement in Investigation 14. It should be noted that the visual could have started from a previous step, i.e. with just one green circle.
In Example 39, we are going to use this as the basic step, so that we have the proof for all positive integer values of $n$.

**Example 39**

Prove by mathematical induction that $1 + 2 + 3 + \ldots + (n-1) + n + (n-1) + \ldots + 3 + 2 + 1 = n^2$.

P($n$): $1 + 2 + 3 + \ldots + (n-1) + n + (n-1) + \ldots + 3 + 2 + 1 = n^2$, $n \in \mathbb{Z}^+$.

When $n = 1$

LHS = 1

RHS = 1

Since LHS = RHS \(\Rightarrow P(1)\) is true

Assume that $P(n)$ is true for some value of $k$, $k \in \mathbb{Z}^+$, i.e.

$1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 = k^2$

When $n = k + 1$ LHS

$= 1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1$

$= 1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 + (k+1) + k$

$= k^2 + (k-1) + k$

$= k^2 + 2k + 1$

$= (k+1)^2$

Since $P(1)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.

We start by making a statement $P(n)$ which we need to prove true for certain values of $n$ (usually positive integers).

This is the basic step.

This is where we make the assumption.

This final statement completes the proof and should always be included.

**Why is the basic step important?**

The principle of mathematical induction is very rigorous, provided that all the steps are used. If the basic step is left out, we can end up with erroneous results. Suppose you were asked whether $10^8$ is a multiple of $7$ and you try using mathematical induction without the basic step.

Here is what you would obtain:

Assume $10^8 = 7a$ for some $n, a \in \mathbb{Z}^+$.

We then move to the inductive step $10^{k+1} = 10 \times 10^k$ and using the assumption would give $10^{k+1} = 10 \times 10^k = 7a = 7(10a)$. Since $a$ is a positive integer, $10a$ is a positive integer also, so $10^8$ is a multiple of $7$ for all positive integers. Of course, we know that this is not true because $10 = 2 \times 5 \Rightarrow 10^8 = 2^8 \times 5^8$ and since $2$ and $5$ are prime numbers $7$ will never divide $10^8$ exactly.

**Incorrect use of the inductive step**

Proof by induction is also not valid if the assumption is not used in the inductive step, as shown below to prove that $11^k - 6$ is a multiple of $5$.

$P(n): 11^k - 6 = 5a$, where $n, a \in \mathbb{Z}^+$.

**Basic step:**

When $n = 1$, LHS = 11 - 6 = 5.

Therefore, the statement is true for $n = 1$.

**Assumption:**

Assume that the statement is true for some $k \in \mathbb{Z}^+$, $k \geq 1$, i.e.

$11^k - 6 = 5b$, where $n, b \in \mathbb{Z}^+$.

**Inductive step:**

When $n = k + 1$:

LHS

$= 11^{k+1} - 6$

$= 11 \times 11^k - 6$

$= (5 \times 6) \times 11^k - 6$ Write 11 as $5 + 6$

$= 5 \times 11^k + 6 \times 11^k - 6$ Distribute $11^k$

$= 5 \times 11^k + 6(11^k - 1)$ 6 is a common factor

$= 5 \times 11^k + 6(11^k - 1)(11^k + 11^{k-2} + 11^{k-4} + \ldots + k + 1)$

$= 5 \times 11^k + 6(11^{k+1} + 11^{k-2} + 11^{k-4} + \ldots + k + 1)$

$= 5(11^k + 12(11^{k-1} + 11^{k-2} + 11^{k-4} + \ldots + k + 1))$

$= 5m$

This is the result required and the mathematics above is correct. However, the assumption was not used to obtain the result. Hence the proof by mathematical induction is incorrect. The correct solution is shown in Example 40.

**Example 40**

Use mathematical induction to prove that $11^k - 6$ is a multiple of 5.

$P(n): 11^k - 6 = 5a$, where $n, a \in \mathbb{Z}^+$

When $n = 1$

Basic step, show statement is true for $n = 1$.

Write the statement that you want to prove.
Prove the following statements using mathematical induction.

Example 41
Prove the following statements using mathematical induction.

a) The sum of the first n terms of an arithmetic sequence with first term \( u_1 \) and common difference \( d \) is given by \( S_n = \frac{n}{2} [2u_1 + (n - 1)d] \).

b) The sum of the first n terms of a geometric sequence with first term \( u_1 \) and common ratio \( r \) is given by \( S_n = \frac{u_1(1 - r^n)}{1 - r} \).

\( S_n = \frac{k+1}{2}(2u_1 + (k + 1)d) \)

Use assumption to obtain result.

\( S_{n+1} = \frac{k+1}{2}(2u_1 + (k + 1) - 1)d) \).

Use algebra to simplify.

Write the final statement.
Use mathematical induction to prove that $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$.

**Proof (p(n)):** $3^{2n} + 7 = 8A$

When $n = 0$,

$LHS = 3^0 + 7 = 8$

So $p(0)$ is true.

Assume that the statement is true for some $k \in \mathbb{N}$, $k \geq 0$.

ie $3^{2k} + 7 = 8A$

$\Rightarrow 3^{2k} = 8A - 7$, $A \in \mathbb{Z}^+$

When $n = k + 1$:

$LHS = 3^{2(k+1)} + 7$

$= 9 \times 3^{2k} + 7$

$= 9(8A - 7) + 7$

$= 72A - 63 + 7$

$= 72A - 56$

$= 8(9A - 7)$

$= 8B$

Since $p(0)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.

**Note that since the statement holds for $n = \mathbb{N}$ we have to start with 0.**

**Assumption.**

Use the assumption.

$(9A - 7) \in \mathbb{Z}^+$, since $A \in \mathbb{Z}^+$.

**Exercise 16**

1 Use the diagrams to answer the questions below.

- Use the binomial theorem and write down a series based on the line divisions.
- Write down a series based on colour.

2 Use mathematical induction to prove the following statements:
   a $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n(n + 1)}{2}^2$
   b $1 - 4 + 9 - 16 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n + 1)}{2}$
   c $\sum_{r=1}^{n} r^2 = 2r^2 - r - 1$
   d $9^n - 1$ is divisible by 8 for all $n \in \mathbb{N}$.
   e $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}$
   f $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.
   g $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
   h $n^3 - n$ is a multiple of 6 for all $n \in \mathbb{Z}^+$.
   i $2n^2 + 3n + 1$, $n \in \mathbb{Z}^+$ is divisible by 7.
   j $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = \frac{n(2n-1)(2n+1)}{3}$
   k $\sum_{r=1}^{n} (r+1) = \frac{n(n+1)(n+2)}{3}$
   l $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

3 Given the statements below, decide how best you can prove or disprove the statement. Some statements can be proved in more than one way; you should attempt both ways and then decide which is the more elegant proof, and why.

a Prove that $(4n + 3)^2 - (4n - 3)^2$ is divisible by 12 for all positive integers $n$.

b $n^2 + 37n + 37$ is a prime number.

c $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)$

b $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1) \times n = \frac{n(n^2 - 1)}{3}$

d $n^3 - n$ is divisible by 3 for all values of $n \in \mathbb{Z}^+$.

**Investigation 15**

Many creates a fun game for practising some mathematics. She arranges 10 cups, numbers them as shown in the diagram on the right, and places one marble just outside cup number 1. She then writes the following instructions.

**Instructions for play:**

- The number of marbles you place in each cup is equal to the number of the cup multiplied by the number of marbles in the previous cup.
- The starting point is cup 1, where you will multiply the number on the cup by the number of marbles outside of cup 1.

1 If you were to follow these instructions, find how many marbles would be placed in:
   a cup number 2
   b cup number 3
   c cup number 5
   d cup number 8.

2 If Mary places another two rows underneath this arrangement, how many cups would there be in total?

3 How can you represent the number of marbles that would be placed in the last cup?

4 Comment on your results.

5 How would you represent the number of marbles in the last cup if there were $n$ cups in total?
Example 43

Find the value of these expressions:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$7!$</td>
<td>b</td>
<td>$3!$</td>
</tr>
<tr>
<td></td>
<td>$7!$</td>
<td></td>
<td>$3!$</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\frac{7!}{5!}$</td>
<td>b</td>
<td>$\frac{3!}{5!}$</td>
</tr>
</tbody>
</table>

You can also find these using technology as follows:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$7! \div 5! = 42$</td>
<td>b</td>
<td>$3! \div 5! = \frac{1}{20}$</td>
</tr>
</tbody>
</table>

Example 44

Simplify the following.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\frac{n(n+1)!}{n!}$</td>
<td>b</td>
<td>$\frac{n! - (n-1)!}{(n+1)!}$</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\frac{n(n+1)!}{n!} = \frac{n(n+1)\times n!}{n!} = n(n+1)$</td>
<td>b</td>
<td>$\frac{n! - (n-1)!}{(n+1)!} = \frac{n(n-1)! - (n-1)!}{(n+1)n(n-1)!} = \frac{n-1}{(n+1)n}$</td>
</tr>
</tbody>
</table>

Rewrite $(n+1)!$ as $(n+1) \times n!$ and $(n+1)!$ as $(n+1) \times n \times (n-1)!$.

Permutations and combinations

Investigation 16a

Angela is creating invitation cards for her birthday party. She has three images which she wants to use on her invitation. She puts the images in a row as shown below.

1. How many arrangements can she choose from?
2. How did you come up with your answer?
3. Angela realizes that she should also include the address, and she still wants to leave the four objects in a line. In how many ways can this be done?
4. She then decides that she does not need to include her address on invitations to her cousins, and realizes that if she makes all invitations individual she will have just enough different invitation cards for all her guests. How many people is she going to invite to her party?
5. What happens if she wants to invite another friend to the party? Explain your answer.
Most probably, when responding to Investigation 16a, all the different arrangements (called permutations) were listed. Here is another way of reasoning out the responses:

Suppose you want to find the total number of arrangements of the letters A, B and C. You have three letters to choose from for the first letter.

Having chosen this first letter, you have two choices for the second letter, and then you are left with only one letter to complete the whole set. In other words, Angela has $3 \times 2 \times 1$ ways of designing the invitation cards for her cousins. You can think of this method as filling boxes as shown here, starting from left to right.

This reasoning can be extended to deduce that the number of ways of arranging $n$ distinct objects can be arranged in a row is $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$

The number of ways of arranging $n$ distinct objects in a row is $n!$

Now suppose that you have five different letters and you want to find the number of possible arrangements of just three of these letters. You can choose the first letter in five ways, the second letter in four ways and the third letter in three ways, giving:

$$5 \times 4 \times 3 = 60$$

Using the same reasoning, you can deduce that the number of ways of arranging three objects chosen out of $n$ distinct objects would be $n \times (n-1) \times (n-2) = n! (n-3)!$. And generalizing even further:

The number of permutations of $r$ objects out of $n$ distinct objects is given by

$$P_r^n = \frac{n!}{(n-r)!}$$

Investigation 16c

Let’s consider what happens if we want to choose three letters out of five and represent these on a chart similar to the one discussed for permutations. If the first letter chosen is A then we have the chart shown here.

1. What do you notice about the colour coded arrangements on the right?
2. What would you expect to notice if the first letter chosen had been B?
3. What if you were to consider all the possible permutations?
4. If the order of choosing the letters is not important, how can you derive the number of combinations?

When the order of arrangements is not relevant we speak about combinations.

The number of ways of choosing $r$ objects from $n$ distinct objects is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

HINT

In some books you may find the notation $\binom{n}{r}$ for $\binom{n}{r}$. In this book we will use the latter which is the notation you will encounter in IB exams.

Example 45

a. In how many ways can the letters of the word candle be arranged?

b. In how many ways can a group of four boys and three girls be arranged?

c. In how many ways can a group of four boys and three girls be arranged if no girls are to be next to each other?

Using a calculator:

| $6!$ | 720 |
| $7!$ | 5040 |
Example 46
How many four-digit numbers can be made using each of the following digits only once?

- a $6p_4 = 4! = 24$
- b $51 = rac{5!}{(8-4)!}$
- c $3 \times 1p_3 = 3 \times 3! = 18$

There are 4! ways of creating a four-digit number with these digits using each digit only once.

The number cannot start with 0. So, there are three ways of choosing the first digit, and the next three digits can be chosen from the three remaining digits which include 0.

Example 47
There are eight boys and five girls who attend the Senior Mathematics Club. Find how many ways the teacher can choose a team of six students to represent the school in a competition if:

- a There are 10 boys and 13 girls who attend the senior mathematics club. Find how many different ways can the books be arranged on the shelf?
- b In how many ways can the books be arranged so that books of the same subject are grouped together?
- c A safe has two dials, one with 26 letters and one with the digits 0 to 9. In order to open the safe, Rose has to choose a code consisting of three distinct letters followed by two distinct digits. Determine how many different safe codes are possible.
- d A delegation of five students is to be selected for a Model United Nations conference. There are 10 boys and 13 girls to choose from.

We cannot have a team with no boys or there are not enough girls to form a team. Using technology:

$$\begin{align*}
13C5 &= 1716 \\
8C3 \times 5C3 &= 560 \\
1716 - 8C5 \times 5C1 - 8C6 \times 5C0 &= 1400
\end{align*}$$

Exercise 1H

1 Copy and complete the table below, simplifying expressions as shown in the first row.

$$\begin{align*}
\frac{7!}{6!} &= 6! [56 - 1] = 39600 \\
\frac{9!}{7!} &= 6! \\
\frac{6!}{5!} &= 6! \\
\frac{9!}{6!} &= 9! \\
\frac{(n+1)!}{n!} - n! &= n! \\
\frac{n!}{(n-1)!} &= n! \\
\frac{(n+1)!}{n!} &= n!
\end{align*}$$

2 Find the value of:

- a $8!$ 
- b $4! \times 5!$ 
- c $10! \times 8!$

Simplify the following:

- a $\frac{(n+1)!}{n!} - n!$ 
- b $\frac{n! + (n+1)!}{n!}$ 
- c $\frac{(n+1)! - 1}{n!}$

4 Show that $\frac{2(n+2)!}{(n+2)!} = \frac{2(2n+1)}{n+1}$

5 Solve for $n \in \mathbb{Z}^+$: $^nC_5 = 66$

6 Solve the equation $16(n-1)! = 5! + (n+1)!$ where $n \in \mathbb{Z}^+$

7 On a bookshelf there are four mathematics books, three science books, two geography books and four history books. The books are all different.

We have to use combinations as the order of choosing is not important. We now need to choose three boys out of eight and three girls out of five.

We cannot have a team with only one girl or with no girls. We also cannot have a team with only one boy.

We have to use combinations as the order of choosing is not important. We now need to choose three boys out of eight and three girls out of five.

We cannot have a team with only one girl or with no girls. We also cannot have a team with only one boy.
11 Graeme is training for a 10 km run. He has six different routes to choose for his training and he trains four times a week. He calculates that he will just manage to run a different set of routes each week leading up to his next race. How many weeks are there before Graeme’s race?

12 A group of 12 people want to go to a concert. They can travel in a small car that takes one driver and one passenger and two cars each taking one driver and four passengers. If there are five drivers in the group, in how many different ways can they travel?

The binomial theorem

Investigation 17

Copy and complete this table by using repeated algebraic multiplication.

<table>
<thead>
<tr>
<th>(1 + x)^n</th>
<th>Constant</th>
<th>Coefficient of x</th>
<th>Coefficient of x^2</th>
<th>Coefficient of x^3</th>
<th>Coefficient of x^4</th>
<th>Coefficient of x^5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 + x)^0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(1 + x)^1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(1 + x)^2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(1 + x)^3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(1 + x)^4</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>(1 + x)^5</td>
<td>1</td>
<td>1</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

1 Comment on any patterns that you recognize.
2 Rearrange the numbers so that rather than forming a right-angled triangle, they form an isosceles triangle with 1 at the top vertex.
3 What new patterns do you notice now?
4 If you extended this pattern, what would you get in the next row?
5 Verify your response to question 4 using algebraic multiplication.

Investigation 18

Consider the expansion (1 + x)^3 = (1 + x)(1 + x)(1 + x).
1 How is the constant term obtained in this expansion?
2 How is the term in x obtained?
3 What about the term in x^2?
4 And the term in x^3?
5 Summarize your responses using mathematical notation.
6 Conceptual: Repeat the process for the expansion (a + x)^7 = (a + x)(a + x)(a + x) ...
7 Write a conjecture for obtaining the expansion of (a + x)^n = (a + x)(a + x)(a + x) ...

Investigation 19

1 Write the expansion of (1 + x)^n using combinations.
2 Find the number of terms in the expansion when:
   a n = 4, 6, 10
   b n is even
   c n = 3, 5, 7
   d n is odd.
3 The pattern found in Investigation 17 is known as Pascal’s triangle. The following properties of this pattern were found:
   a There is a line of symmetry going down the middle of the numbers.
   b Each row starts and ends with 1 and each of the other numbers is the sum of the two numbers above it to either side.
   c The expansion of (1 + x)^n can be written as follows:
      (1 + x)^n = C(0, n)x^n + C(1, n)x^{n-1} + C(2, n)x^{n-2} + ... + C(n, n)x^0
      a Use this expansion and the one for (1 + x)^n to verify the two properties above.
      b Write the expansions for (1 + x)^n and (1 + x)^{n+1}.
      c Use the same method as in part a to show that the two properties above hold.
      d Conceptual: How can you explain the patterns in Pascal’s triangle by considering the general expansion of the binomial expansion?

Example 48

Find the values of a, b and c in the following identities:

a (1 + 2x)^3 = 1 + ax + bx^2 + cx^3 + ... - 32x^3
b \( \left( 1 + \frac{1}{2} \right)^3 = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + ... + \frac{1}{8}x^3 \)

\( (2 + ax)^3 = b + 224x + ax^2 + 70ax^3 + ... \)

a (1 - 2x)^3 = C(0, -2x)^3 + C(1, -2x)^3 + C(2, -2x)^3 + ... + C(3, -2x)^3
b (1 + 5x)^3 = 1 + 15x + 45x^2 + 225x^3

You should know the first five rows of Pascal’s triangle:

\( (1 - 2x)^3 = (1 + (2x))^3 \)
b \[
\left(1 + \frac{x}{3}\right)^5 = 1 + 5 \frac{x}{3} + 10 \left(\frac{x}{3}\right)^2 + 10 \left(\frac{x}{3}\right)^3 + 5 \left(\frac{x}{3}\right)^4 + \left(\frac{x}{3}\right)^5
\]
\[
\Rightarrow a = 8
\]
\[
\left(1 + \frac{x}{3}\right)^5 = 1 + 5 \frac{x}{3} + 10 \left(\frac{x}{3}\right)^2 + 10 \left(\frac{x}{3}\right)^3 + 5 \left(\frac{x}{3}\right)^4 + \left(\frac{x}{3}\right)^5
\]
\[
= 1 + \frac{81}{16} \frac{x}{3} + \frac{81}{216} \frac{x^2}{9} + \frac{28}{9} \frac{x^3}{27} + \frac{5}{27} \frac{x^4}{81} + \frac{x^5}{243}
\]
\[
\Rightarrow b = \frac{8}{3} \text{ and } c = \frac{28}{9}
\]

c \[
(2 + ax)^7 = 2^7 + 7 \cdot 2^6 \cdot (ax) + 21 \cdot 2^5 \cdot (ax)^2 + 35 \cdot 2^4 \cdot (ax)^3 + 35 \cdot 2^3 \cdot (ax)^4 + \ldots
\]
\[
= 128 + \frac{7}{116} \cdot 2^2 \cdot ax + \frac{7}{116} \cdot 2^2 \cdot ax + \frac{7}{116} \cdot 2^2 \cdot ax + \ldots
\]
\[
= 128 + 448a \cdot x + 672a^2 \cdot x^2 + 560a_3 \cdot x^3 + \ldots
\]
\[
\Rightarrow a = \frac{1}{2} \text{ and } c = 168
\]

Example 49
Find the coefficient of \(x^3y^3\) in the expansion of \((x + 3y)^6\).

The general term in the expansion of \((x + 3y)^6\) is given by \(\binom{6}{r} x^{6-r} (3y)^r\).

We want \(\binom{6}{r} x^{6-r} (3y)^r = Ax^2y^3\) \(\Rightarrow r = 3\).

Then \(A = \binom{6}{3} \cdot 3^3 = \frac{6!}{3!3!} \cdot 3^3 = 27 \cdot 540\).

Example 50
Use the binomial theorem to expand \((2x + 3y)^5\). Hence find the value of 2.035 correct to 5 decimal places.

\[
(2x + 3y)^5 = 32x^5 + 240x^3y^2 + 720xy^4 + 1080y^5 + 810y^4 + 243y^5
\]

When \(x = 1, y = 0.01\) we obtain

\[
(2.03)^5 = 32.240 + 0.0720 + 0.001080 + 0.00000810 + 0.000000243
\]

\[
= 32.47309 \text{ (to 5 decimal places)}
\]

Example 51
Find the term independent of \(x\) in the expansion of \(\left(x - \frac{1}{2x}\right)^7\).

The general term in the expansion of \(\left(x - \frac{1}{2x}\right)^7\) is given by:

\[
\binom{7}{r} x^{7-r} \left(-\frac{1}{2x}\right)^r = \binom{7}{r} \left(-\frac{1}{2}\right)^r x^{7-2r}
\]

For the term independent of \(x\):

\[
12 - 2r = 0 \Rightarrow r = 6
\]

\[
\binom{7}{6} \left(-\frac{1}{2}\right)^6 x^1 = \binom{7}{6} \left(-\frac{1}{2}\right)^6
\]

\[
= \frac{7!}{6!1!} \times \left(-\frac{1}{2}\right)^6 = \frac{7}{1} \times \frac{1}{64} = \frac{7}{64}
\]

Give your answer as an exact fraction.

Exercise 11
1. Write the first four terms in the binomial expansion of:
   a. \(\left(1 - \frac{x}{3}\right)^6\)
   b. \(\left(1 + \frac{x}{2}\right)^7\)
   c. \(\left(2 + \frac{x}{3}\right)^7\)

2. In each of the following binomial expressions, write down the required term.
   a. fifth term of \((a - 2b)^{10}\)
   b. third term of \((a + \frac{1}{a})^{10}\)
   c. fourth term of \((2 - \frac{x}{y})^8\)

3. Find the term independent of \(x\) in the expansion of \(\left(x - \frac{2}{x}\right)^{10}\).

4. Use the binomial theorem to expand \(2 - \frac{x}{2}\). Hence find the value of \((1.99)^4\) correct to 5 decimal places.

5. Find the term in \(x^6\) in the expansion of \(\left(x^2 + \frac{1}{y}ight)^8\).

6. a. Expand \(\left(x - \frac{2}{x}\right)^3\)
   b. Find the coefficient of \(x^3y^2\) in the expansion of \((2x + y)(x + \frac{y}{x})^3\).
Generalization of the binomial expansion

It was around 1665 that Isaac Newton generalized the binomial theorem to allow for negative and fractional exponents. Let’s try to examine this using some facts which were established earlier in this chapter.

Consider the geometric series $1 + x + x^2 + x^3 + \ldots$, where $x$ is not equal to 0. For which values of $x$ does this series converge?

What is the sum to infinity for this series when it converges?

We can write the answers to these two questions as follows:

For $-1 < x < 1$, $S = \frac{1}{1-x}$

In other words:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

If we want to expand $(1-x)^{-1}$ we could say that this is equivalent to

$$((1-x)^{-1})^2 = (1-x)^{-2}$$

$$= (1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$$

$$= 1 + x + x^2 + x^3 + \ldots$$

... multiplying the terms in the left bracket by 1

$$+ x^2 + x^3 + x^4 + \ldots$$

... multiplying the terms in the left bracket by $x$

$$\ldots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$\Rightarrow (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Similarly, we can repeat the process to obtain the expansion of $(1-x)^{-3}$:

$$((1-x)^{-1})^3 = (1-x)^{-3}$$

$$= (1 + x + x^2 + x^3 + \ldots)^3$$

$$= 1 + 3x + 6x^2 + 10x^3 + \ldots$$

Newton generalized this result for negative and rational exponents of the binomial theorem as follows:

The binomial expansion for $(1-x)^{-1}$ for $x \in \mathbb{Z}$ and $-1 < x < 1$ is given by the infinite series:

$$(1-x)^{-1} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

For which values of $x$ does the binomial theorem as follows:

Newton generalized this result for negative and fractional exponents of the binomial theorem as follows:

The binomial expansion for $(1 + x)^n$ for $n \in \mathbb{Z}$ and $-1 < x < 1$ is given by the infinite series:

$$(1+x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

For which values of $x$ does the binomial theorem as follows:

Investigation 20

1. **Factual** Show that the generalized form for negative integer powers given above can be written as

$$(1-x)^k = \sum_{r=0}^{\infty} \binom{k}{r} x^r$$

The table below includes the results obtained above.

<table>
<thead>
<tr>
<th>$(1-x)^k$</th>
<th>Constant</th>
<th>Coefficient of $x^1$</th>
<th>Coefficient of $x^2$</th>
<th>Coefficient of $x^3$</th>
<th>Coefficient of $x^4$</th>
<th>Coefficient of $x^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1-x)^1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$(1-x)^2$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$(1-x)^3$</td>
<td>1</td>
<td>-3</td>
<td>6</td>
<td>-10</td>
<td>20</td>
<td>-15</td>
</tr>
<tr>
<td>$(1-x)^4$</td>
<td>1</td>
<td>-6</td>
<td>12</td>
<td>-20</td>
<td>30</td>
<td>-30</td>
</tr>
<tr>
<td>$(1-x)^5$</td>
<td>1</td>
<td>-15</td>
<td>30</td>
<td>-50</td>
<td>70</td>
<td>-70</td>
</tr>
</tbody>
</table>

2. Use Newton’s generalization to verify that the coefficients shown in the table are correct.

3. Apply Newton’s generalization to copy and complete the table.

4. **Factual** Show that the generalized form of the binomial theorem for fractional powers can be written as

$$(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

5. Use the generalization to find the expansion of $\sqrt{1+x}$ and $\sqrt{1-x}$.

**Example 52**

Expand the following up to the term in $x^3$.

**a** $\sqrt{1+2x}$, for $|x| < \frac{1}{2}$

**b** $\sqrt{1-3x}$, for $|x| < \frac{1}{3}$

Using

$$(1 + x)^r = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \ldots$$

Simplify.
b \[\frac{2}{(1-x)^3} = 2[1-3x]^{-3}\]
\[= \frac{1}{(1-x)^3} = 1 + 3x + \frac{9x^2 + 27x^3 + \cdots}{3!} + \frac{21x^4 + \cdots}{3!} + \cdots\]
Using
\[(1-x)^{-1} = 1 + nx + \frac{n(n+1)x^2}{2!} + \frac{n(n+1)(n+2)x^3}{3!} + \cdots,\]
for all \(n \in \mathbb{Z}^+\) and \(|x| < \frac{1}{3}\).

Example 53
Use the binomial expansion to show that \(\sqrt{1+x} = 1 + \frac{1}{2} x^2\), \(|x| < 1\).

\[\sqrt{1+x} = 1 + \left\{1 + \left(\frac{1}{2}\right)x^2 + \cdots\right\} - x^3 = 1 + \frac{1}{2} x^2 + \cdots\]
Using
\[(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2} x^2 + \cdots,\]
we only need the expansion until the term in \(x^2\).

\[(1-x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} = \left(1 - x + \frac{1}{2} x^2 + \cdots\right) = 1 + \frac{1}{2} x^2 + \cdots\]

To what extent do you agree with this quote?

E. Mach

"Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting."

Exercise 1J

1. Expand the following up to the term in \(x^2\), given that \(|x| < \frac{1}{2}\).
   a. \(\frac{1}{1+x}\)
   b. \(\frac{1}{1-2x}\)
   c. \(\frac{2}{1-2x}\)
   d. \(\frac{2}{1-3x}\)

2. Find the first two terms of each of the following expansions where \(|x| < \frac{1}{10}\).
   a. \(\sqrt{1+2x}\)
   b. \((1+x)^{\frac{1}{2}}\)
   c. \((1-3x)^{\frac{1}{2}}\)
   d. \(2(1+x)^{\frac{1}{2}}\)

3. Show that \(\sqrt{1+x} = 1 - x + \frac{x^2 - x^3}{2}\), where \(|x| < 1\).

4. Show that \(\frac{x}{(1+x)^3} = x - 2x^2 + 3x^3 - 4x^4\).

5. Find the binomial expansion of \((2-3x)^{\frac{1}{4}}\), \(|x| < \frac{1}{2}\).

6. a. Find the first four terms of the binomial expansion of \(\sqrt[4]{1-4x}\), \(|x| < \frac{1}{4}\).

b. Show that the exact value of \(\sqrt[4]{1-4x}\) when \(x = \frac{1}{100}\) is \(\frac{\sqrt[4]{5}}{5}\).

c. Hence, determine \(\sqrt[4]{5}\) to 5 decimal places.

7. a. Find the first three terms of the binomial expansion of \(\sqrt[3]{1-2x}\), \(|x| < \frac{1}{2}\).

b. Hence or otherwise, obtain the expansion of \(\sqrt[3]{2+3x}\), \(|x| < \frac{1}{2}\).

Chapter summary

- Sequences may be finite or infinite.
- If the difference between two consecutive numbers in a sequence is constant then it is an arithmetic sequence or an arithmetic progression. The constant difference is called the common difference and is denoted by \(d\).
- An arithmetic sequence with first term \(u_1\) and common difference \(d\) has general term \(u_n = u_1 + (n-1)d\).
- The sum of a finite arithmetic series is given by \(S_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}[u_1 + u_n]\) where \(n\) is the number of terms in the series, \(u_1\) is the first term, \(d\) is the common difference and \(u_n\) is the last term.
If the ratio of two consecutive terms in a sequence is constant then it is a geometric sequence or a geometric progression. We call the constant ratio the common ratio and denote it by \( r \).

A geometric sequence with first term \( u_1 \) and common ratio \( r \) has general term \( u_n = u_1 r^{n-1} \), \( r \neq 0 \), \( -1, u_1 \neq 0 \).

The sum of a finite geometric series is given by

\[
S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1
\]

where \( n \) is the number of terms, \( u_1 \) is the first term and \( r \) is the common ratio.

The sum of \( n \) terms of a geometric series is

\[
S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1.
\]

When \( -1 < r < 1 \), \( r^n \) approaches zero for very large values of \( n \). The series therefore converges to a finite sum given by \( S = \frac{u_1}{1-r} \).

A proof in mathematics often consists of a logical set of steps that validates the truth of a general statement beyond any doubt.

A direct proof is a way of showing the truth of a given statement by constructing a series of reasoned connected established facts. In a direct proof the following steps are used:

1. Identify the given statement.
2. Use axioms, theorems, etc., to make deductions that prove the conclusion of your statement to be true.

When setting out a proof by contradiction you follow the following steps:

1. Identify what is being implied by the statement.
2. Assume that the implication is false.
3. Use axioms, theorems, etc., to arrive at a contradiction.
4. This proves that the original statement must be true.

A counterexample, or counterclaim, is an acceptable "proof" of the fact that a given statement is false.

\[ n! = n \times (n-1)! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \]

The number of ways of arranging \( r \) distinct objects in a row is \( r! \).

The number of permutations of \( r \) objects out of \( n \) distinct objects is given by

\[
P(n,r) = \frac{n!}{(n-r)!}
\]

The number of ways of choosing \( r \) objects from \( n \) distinct objects is \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

The binomial expansion for \((1-x)^m\) for \( m \in \mathbb{Z}^+ \) and \(-1 < x < 1\) is given by the infinite series:

\[
(1-x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots
\]

The binomial expansion for \((1+x)^m\) for \( a = \frac{p}{q} \in \mathbb{Q} \) and \(-1 < x < 1\) is given by the infinite series:

\[
(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots
\]

Developing inquiry skills

<To come>

Chapter review

1. Show that there are two geometric sequences such that the second term is 9 and the sum of the first three terms is 91. Write the fourth term of each sequence.

2. Find the sum of the series 1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 + 11 + 13 + 15 + 16 + 17 + \ldots + 64.

3. Three numbers \( a \), \( b \) and \( c \) form an arithmetic sequence. The numbers \( b \), \( c \) and \( a \) form a geometric sequence. Find the three numbers given that they add up to 36.

4. a) Prove the identity

\[
\frac{1}{1+x} + \frac{1}{3+2x} = \frac{x+2}{2x^2+5x+3}
\]

b) Hence, use the binomial expansion to find the first four terms of the expansion of

\[ \frac{x+2}{2x^2+5x+3} \]

5. Prove the following identities:

a) \( \binom{2}{0} = \binom{2}{1} + n \)

b) \( \binom{2}{0} \times \binom{3}{2} = \binom{5}{2} \times \binom{2}{2} \)
6. Show that \(c_0 + 3 \times c_1 + 3^2 \times c_2 + \ldots + 3^n = \sum_{i=0}^{n} 3^i = 2^n\).

7. Prove by contradiction that no two integers \(a\) and \(b\) can be found such that \(14a + 7b = 1\).

8. Prove by contradiction that if \(x = 3\) then \(5x - 7 = 13\).

9. Give a counterexample to prove that each of the following statements is false:
   a. If \(a^2 - b^2 = 0\) then \(a = b\) and \(b = -a\).
   b. \(y^2 + 2\) is prime for all \(n \in \mathbb{Z}\).
   c. \(\sqrt{2n} - 1\) is irrational for all \(n \in \mathbb{Z}\).
   d. \(2n - 1\) is prime for all \(n \in \mathbb{Z}\).

10. Prove by mathematical induction that \((1 \times 1) \times (2 \times 2) \times (3 \times 3) \times (4 \times 4) \times \ldots \times (n^2 \times n^2) = (n^n)^{n^{n-1}}\).

11. Use mathematical induction to prove that \(n^3 + 2n\) is a multiple of 3.

12. Use mathematical induction to prove the following statements:
   a. \(\sum_{r=0}^{n} r = \frac{n(n + 1)}{2}\)
   b. \(\sum_{r=0}^{n} r^2 = \frac{n(n + 1)(2n + 1)}{6}\)
   c. \(\sum_{r=0}^{n} r^3 = \frac{n^2(n + 1)^2}{4}\)

Hence prove that
   \[\sum_{r=0}^{n} (r + 1)(r + 2) = \frac{n(n + 1)(2n + 3)}{4}\]

13. a. In how many ways can the letters of the word harmonics be arranged?
   b. Determine how many numbers bigger than 30,000, less than 9,999,999 and divisible by 5 can be formed using the digits 0, 1, 2, 3, 5, 8 and 9.
   c. In how many ways can a committee of five people be selected from seven men and four women, so that there is at least one male and one female and there are more women than men on the committee?

14. Let \(a = x + y\) and \(b = x - y\).
   a. Write \(a^2 - b^2\) in terms of \(x\) and \(y\) and hence show that \(a^2 - b^2 = (a - b)(a + b)\).
   b. Use the binomial theorem to write \(a^4\) and \(b^4\) in terms of \(x\) and \(y\).
   c. Use your results to part b to show that \(a^4 - b^4 = (a - b)(a^3 + ab + b^3)\).
   d. Use the binomial theorem to write \(a^4\) and \(b^4\) in terms of \(x\) and \(y\) and use your result to factorize \(a^4 - b^4\).
   e. Use your results to make a conjecture for the factors of \(a^n - b^n\).
   f. Prove your conjecture using mathematical induction.

15. Given that the coefficients of \(x^{-1}\), \(x^0\) and \(x^1\) in the expansion of \((1 + x)^n\) are in arithmetic sequence, show that \(n^2 + 4n^2 - 2n(4n + 1) = 0\).

Hence find three consecutive coefficients of the expansion of \((1 + x)^{14}\) which form an arithmetic sequence.

16. Given that
   \[2 + x - 3x^2 = \frac{A}{1 - 2x} + \frac{B}{1 - 3x} + \frac{C}{1 - x}\]
   determine the values of \(A, B\) and \(C\).

Hence use the binomial theorem to find the expansion of \(2 + x - 3x^2\) in ascending powers of \(x\) up to and including the term in \(x^3\).

17. Find the coefficient of the term in \(x^3\) in the binomial expansion of \((3 + x)(1 - 2x)^2\).

18. The coefficient of \(x^n\) in the binomial expansion of \((1 + 3x)^7\) where \(n \in \mathbb{N}\) is 495.

Determine the possible values of \(n\).

19. Find the value of \(\sum_{n=1}^{10} (1.6^n - 12n + 1)\), giving your answer correct to 1 decimal place.

20. Prove the binomial coefficient identity
   \[\binom{n}{k} = \frac{n-1}{k-1}\]

(6 marks)

21. Find the sum of all integers between 500 and 1400 (inclusive) that are not divisible by 7.

22. Prove by contradiction that for all \(n \in \mathbb{Z}\), if \(n^3 + 3\) is odd, then \(n\) is even.

(7 marks)

23. Prove, by mathematical induction, that \(5^{n+1} + 1\) is divisible by 6 for all \(n \in \mathbb{N}\).

(8 marks)

24. a. Find the first four terms, in ascending powers of \(x\), of the binomial expansion of \(\sqrt[4]{1 - x}\). (4 marks)
   b. Use your answer to part a) to find an approximation for \(\sqrt[4]{3}\) to six decimal places. You must show all your working.

(5 marks)

25. Seven women and two men are chosen to sit in a row and have their photographs taken.
   a. How many different ways can they be arranged? (1 mark)
   b. How many ways can they be arranged if the men must sit together? (2 marks)
   c. How many ways can they be arranged if the men must sit apart? (2 marks)
   d. How many ways can they be arranged if there must be at least two women separating the men? (3 marks)
The Towers of Hanoi

The problem

The aim of the Towers of Hanoi problem is to move all the disks from peg A to peg C following these rules:

1. Move only one disk at a time.
2. A larger disk may not be placed on top of a smaller disk.
3. All disks, except the one being moved, must be on a peg.

For 64 disks, what is the minimum number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for 3 and 4 disks.

What is the minimum number of moves needed in each case?

Solving the problem for 64 disks would be very time consuming, so you need to look for a rule for \( n \) disks that you can then apply to the problem with 64 disks.

Try and test a rule

Assume the minimum number of moves follows an arithmetic sequence.

Use the minimum number of moves for 3 and 4 disks to predict the minimum number of moves for 5 disks.

Check your prediction using the simulator.

Does the minimum number of moves follow an arithmetic sequence?

Find more results

Use the simulator to write down the number of moves when \( n = 1 \) and \( n = 2 \).

Organize your results so far in a table.

Look for a pattern. If necessary, extend your table to more values of \( n \).

Try a formula

Return to the problem with 4 disks.

Consider this image of a partial solution to the problem. The large disk on peg A has not yet been moved.

Consider your previous answers.

What is the minimum possible number of moves made so far?

How many moves would it then take to move the largest disk from peg A to peg C?

When the large disk is on peg C, how many moves would it then take to move the 3 smaller disks from peg B to peg C?

How many total moves are therefore needed to complete this puzzle?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete this puzzle with \( n \) disks.

This is an example of a recursive formula. What does that mean?

How can you check if your recursive formula works?

What is the problem with a recursive formula?

Try another formula

You can also try to solve the problem by finding an explicit formula that does not depend on you already knowing the previous minimum number.

You already know that the relationship is not arithmetic.

How can you tell that the relationship is not geometric?

Look for a pattern for the minimum number of moves in the table you constructed previously.

Hence write down a formula for the minimum number of moves in terms of \( n \).

How does an explicit formula differ from a recursive formula?

Use your explicit formula to solve the problem with 64 disks.

Extension

- What would a solution look like for 4 pegs?
  Does the problem become harder or easier?

- Research the ‘Bicolor’ and ‘Magnetic’ versions of the Towers of Hanoi puzzle.

- Can you find an explicit formula for other recursive formulae? (e.g. Fibonacci)