Chapter 1: Standard functions and techniques

1.1. (a) \( y = x^4 \) for \(-1.5 \leq x \leq 1\):

(b) \( y = x(1-x) \) for \(-1 \leq x \leq 2\):

(c) \( y = 1 + x + x^2 \) for \(|x - 1| \leq 2\):

(d) \( y = |x - 1| \) for \(-3 \leq x \leq 3\):
Problem 1.1e

Figure 5: Problem 1.1e

Problem 1.1f

Figure 6: Problem 1.1f

\[ (e) \quad y = |x| + |x - 3| + |x + 2| \] for \(-3 \leq x \leq 4\):
\[ (f) \quad y = |x| - 1 \] for \(-2 \leq x \leq 2\):
\[ (g) \quad y = \sqrt{x^2 + 1} \] for \(|x| \leq 2\):

Problem 1.1g

Figure 7: Problem 1.1g

Problem 1.2

(a) \( y = -2x + 3 \);
(b) \( y = 1 \);
(c) \( y = \frac{2}{3}x - \frac{1}{4} \). The intersections occur at \( A : (2, 1) \), \( B : \left(\frac{5}{4}, \frac{1}{2}\right) \), \( C : (1, 1) \). The side lengths are: \( AB = \frac{1}{4}\sqrt{13} \), \( BC = \frac{1}{2}\sqrt{5} \), \( CA = 1 \).

Problem 1.3

(a) Slope is 1, and the line cuts the axes at \((0, -1)\) and \((1, 0)\).
(b) Slope is \(\frac{1}{3}\), and the line cuts the axes at \((0, -\frac{2}{3})\) and \((2, 0)\).
(c) Slope is \(-\frac{2}{5}\), and the line cuts the axes at \((0, -\frac{2}{5})\) and \((2, 0)\).

Problem 1.4

(a) \( y = x + 1 \);
(b) \( y = -2x - 4 \);
(c) \( y = 0.5x - 0.5 \);
(d) \( y = 3x - 1 \);
(e) the slope of the line must be \(-\frac{1}{2} \): \( y = -\frac{1}{4}x + \frac{11}{4} \).

Problem 1.5

The products of the slopes in each case must be \(-1\). The slopes are: \( (a) \ -\frac{3}{2} \) and \( \frac{4}{7} \); \( (b) \ 2 \) and \( -\frac{1}{2} \); \( (c) \ 2 \) and \( -\frac{1}{2} \); \( (d) \ 1 \) and \(-1\).

Problem 1.6

At the point of intersection, \( x + y + 1 = 0 \) and \( 2x - 3y - 2 = 0 \), so the line
\[ (x + y + 1) + \alpha(2x - 3y - 2) = 0 \]
must pass through this point, which has coordinates \((-\frac{1}{2}, -\frac{4}{5})\). The straight line joining this point to \((1, 1)\) is\
\[2y = 3x - 1,\]
with \(\alpha = \frac{1}{2}\).

1.7. (a) Centre at \((0, 0)\), radius 3; (b) centre at \((1, 0)\), radius 2; (c) centre at \((1, 1)\), radius \(\sqrt{23}\); (d) centre at \((\frac{1}{2}, -\frac{1}{2})\), radius \(\frac{1}{2}\sqrt{11}\).

1.8. \((x - 1)^2 + (y + 2)^2 = 9\).

1.9. Eliminate one of the variables in each case and solve the resulting quadratic equation.
(a) \((2, 2)\) and \((2, -2)\);
(b) Eliminate \(y\), so that \(x\) satisfies the equation\
\[x^2 + (2x + 1)^2 - 2x + 2(2x + 1) - 4 = 0, \text{ or } 5x^2 + 6x - 1 = 0.\]
The points of intersection are\
\[(\frac{1}{5}(-3 - \sqrt{14}, -1 - 2\sqrt{14}) \text{ and } (\frac{1}{5}(-3 + \sqrt{14}, -1 + 2\sqrt{14})\]
(c) \((\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})\), one point only since the line is tangential to the circle.

1.10. To three decimal places, the distances of the points from the origin are
\[1.060, \ 0.993, \ 1.011, \ 0.896, \ 1.124.\]
The average value of these distances is \(r = 1.017\). The equation of the circle is\
\[x^2 + y^2 = r^2 = 1.034.\]

1.11. (a) \(x = H(t + 1) - H(t - 1)\).

(b) \(x = \text{sgn}(1 + t) + \text{sgn}(1 - t)\).

\[\text{Figure 9: Problem 1.11a}\]

(c) \(x = tH(t - 1)\).
(d) \((t^2 - 1)[\text{sgn}(t + 1) + \text{sgn}(1 - t)]\).

1.12. (a) \(f(t) = H(2 - t) + H(t + 1) - 1\); (b) \(f(t) = 2tH(t)\);
(d) \(f(t) = (3 - t)H(3 - t) + (t - 2)H(2 - t) + (t - 1)H(1 - t) - tH(-t)\).
1.13. (a) $\frac{1}{2}\pi$ radians; (b) $\frac{3}{2}\pi$ radians.

1.14. (a) $1/\sqrt{2}$; (b) 1; (c) 0; (d) $-1/\sqrt{2}$; (e) $\sqrt{3}/2$; (f) $-\sqrt{3}/2$; (g) $\sqrt{3}/2$; (h) $-\sqrt{3}/2$.

1.15. (a) Using the identity $\cos^2 B = \frac{1}{2}(1 + \cos 2B)$

$$\cos^4 A = \frac{1}{4}(1 + \cos 2A)^2 = \frac{1}{4}(1 + 2 \cos 2A + \cos^2 2A)$$
$$= \frac{1}{4}(1 + 2 \cos 2A + \frac{1}{2}(1 + \cos 4A)) = \frac{1}{8}(3 + 4 \cos 2A + \cos 4A).$$

(b) Use the identities $\sin^2 B = \frac{1}{2}(1 - \cos 2B)$ and $\cos^2 2B = \frac{1}{2}(1 + \cos 4B)$:

$$\sin^4 A = \frac{1}{4}(1 - 2 \cos 2A + \cos^2 2A) = \frac{3}{8} - \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A.$$

1.16. (a) $\cos(x + \frac{1}{4}\pi) = \cos x \cos \frac{1}{4}\pi - \sin x \sin \frac{1}{4}\pi = \sin x$; (b) $\cos x$; (c) $-\cos x$; (d) $-\cos x$ (for both); (e) $\sin x$ (for both).

1.17. (a)

$$\cos x + \cos y = \cos \left[ \frac{1}{2}(x+y) \right] + \cos \left[ \frac{1}{2}(x-y) \right]$$
$$= \cos \left[ \frac{1}{2}(x+y) \right] \cos \left[ \frac{1}{2}(x-y) \right] - \sin \left[ \frac{1}{2}(x+y) \right] \sin \left[ \frac{1}{2}(x-y) \right]$$
$$= 2 \cos \left[ \frac{1}{2}(x+y) \right] \cos \left[ \frac{1}{2}(x-y) \right].$$

(b) $\sin x - \sin y = 2 \sin \left[ \frac{1}{2}(x+y) \right] \cos \left[ \frac{1}{2}(x-y) \right].$

(c) $\cos x - \cos y = -2 \sin \left[ \frac{1}{2}(x+y) \right] \sin \left[ \frac{1}{2}(x-y) \right].$

1.18. (a) $x = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$;

(b) $x = \frac{1}{2}(2n+1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$;

(c) $x = 2n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$;

(d) $x = \frac{1}{2}(2n+1)$, $(n = 0, \pm 1, \pm 2, \ldots)$;

(e) $x = \frac{1}{2}n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$;

(f) $x = 2n$, $(n = 0, \pm 1, \pm 2, \ldots)$. 

4
1.19. \[ \begin{array}{cccc} \text{amplitude} & \text{angular frequency} & \text{period} & \text{phase} \\ (a) & 2 & 0.2 & 10\pi & 3.2 \\ (b) & 1.5 & 0.2 & 10\pi & -0.48 \\ (c) & 3.87 & 0.2 & 10\pi & -0.135 \\ (d) & 1 & 1 & 2\pi & \pi \end{array} \]

1.20. (a) \( F(x) = \frac{1}{2}\sqrt{x} \); (b) \( F(x) = \frac{1}{2}(x - 3) \); (c) \( F(x) = \frac{1}{2}\arcsin x \); (d) \( F(x) = \arcsin \left( \frac{1}{2}x \right) \); (e) \( F(x) = \arcsin x \); (f) \( F(x) = \arcsin \left( \frac{1}{2}\arcsin x \right) \); (g) \( F(x) = x^{-1} \); (h) \( F(x) = -\frac{1}{2} + \sqrt{(x + \frac{1}{4})} \).

1.21. The graph shows \( y = x^3 - x + 1 \) (the dashed curve) and its inverse.

![Figure 13: Problem 1.21](image)

1.22. (a) \( x = \frac{1}{2}\ln 3 \); (b) \( x = \frac{1}{4}e^2 \); (c) \( x = e^{-1} \); (d) \( x = -\frac{1}{2}\ln 3 \); (e) the equation is the same as \( (e^x - 1)^2 = 0 \); hence \( x = 0 \); (f) \( x = 2 \); (g) \( x = 2/17 \); (h) \( x = \pm\sqrt{2} \); (i) \( x = \pm\sqrt{1 + e^2} \); (j) \( x = (\ln 3)/(\ln 2) \); (k) \( x = -(\ln 2)/(2\ln 3) \); (l) \( x = \frac{1}{4}\ln[4 + \sqrt{17}] \); (m) no solutions.

1.23. \( 2^x = e^{x\ln 2} \).

1.24. Consider two values of \( x \), say \( x_1 \) and \( x_2 \), where \( x_1 > x_2 \). Then if \( 10^{x_1} = 2 \times 10^{x_2} \), it follows that

\[ 10^{x_1 - x_2} = 2, \text{ or } x_1 - x_2 = \frac{\ln 2}{\ln 10} \]

an interval which is independent of \( x_1 \) and \( x_2 \).

1.25. (a) \( (x - 1)^2 + y^2 \leq 9 \).

![Figure 14: Problem 1.25a](image)

(b) \( x \geq 0, y \geq 0 \), and \( x + y \leq 1 \).
(c) \( (x^2/4) + (y^2/9) \leq 1 \).
(d) \( x^2 + y^2 \leq 1 \) and \( x \geq 0 \).
(e) \( |x| + |y| \leq 1 \).

1.26. Let

\[ x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} \]
where $-1 < x < 1$. Hence

$$(1 - x)e^{2y} = 1 + x$$

so that $y = \frac{1}{2}\ln \left(1 + \frac{x}{1-x}\right)$

as required.

1.27. From triangle ABC

$$AC = AB\sin \theta + \sqrt{(BC^2 - AB^2\cos^2 \theta)}$$

$$= 2.5[\sin \omega t + \sqrt{(4 - \cos^2 \omega t)}] \text{ cm},$$

where $\theta = \omega t$. The angular frequency $\omega = 400\pi/3$.

1.28. $x = 5\cos(\omega t - 0.927)$. The amplitude $c = 5$ and the phase angle is $\phi = -0.927$.

1.29. $f(0) = 2$ implies $C = 2$ and $f(1) = 0.5$ implies $0.5 = Ce^{-\alpha} = 2e^{-\alpha}$. Hence $\alpha = \ln 4$. Also $f(2) = \frac{4}{5}$.

1.30. The tidal period is $2\pi/0.5 = 12.57$ hours. We require the times when the depth is 2m in one period, which are given by solutions of

$$2 = 5 + 4.5\sin 0.5t$$

so that $\sin 0.5t = -\frac{2}{3}$.
Two consecutive times are 11.11 hours and 7.74 hours. Hence the yacht can float free for 9.20 hours in each tidal period. The yacht floats when \( \sin 0.5t > -\frac{2}{3}\). It is helpful to sketch \( y = \sin 0.5t \) and \( y = -\frac{2}{3} \) and plot their intersections.

**1.31.** (a) The cardioid \( r = 0.5(1 + \cos \theta)\):

(b) The folium \( r = (4 \sin^2 \theta - 1) \cos \theta \)

(c) \( r = \sin 2\theta\)

(d) The Archimedean spiral \( r = 0.04\theta \)

(e) The equiangular spiral \( r = 0.1e^{0.1\theta} \)

**1.32.** (a) \( \text{sgn} (\sin x) \):

(b) \( \text{sgn} \cos 2x \)
(c) $H(x) \sin x$:
(d) $\sin^2 x$:
(e) $|\sin x|$:
(f) $\sin |x|$:
(g) $H(x - \pi) \sin x$:

1.33. Let the points be $A : (-7, 3), B : (1, -3)$ and $C : (4, 1)$. The slope of $AB$ is $-\frac{3}{4}$ and the slope of $BC$ is $\frac{4}{3}$; the product of the slopes is $-1$ which means that $ABC$ is a right angle. Let $D$ be the fourth vertex. Then the equations of the lines $AD$ and $DC$ are

\[ y - 3 = \frac{4}{3}(x + 7) \quad \text{and} \quad y - 1 = -\frac{3}{4}(x - 4), \]

or

\[ 3y - 4x - 37 = 0 \quad \text{and} \quad 4y + 3x - 16 = 0. \]

These lines intersect at the point $D : (-4, 7)$.

There is a general formula buried here, if you notice that the coordinates of $D$ are $(-7 + 4 - 1, 3 + 1 - (-3))$.

1.34. (a) periodic, period $\frac{1}{2}\pi$; (b) periodic, period $2\pi$; (c) periodic, period $2\pi$; (d) not periodic; (e) periodic, period $2\pi$; (f) periodic, period $\pi$; (g) not periodic; (h) periodic, period $\pi$; (i) periodic, period $\pi$; (j) periodic, period $8\pi$; (k) periodic, period $\frac{\pi}{3}$, since $\sin 3t$ has period $\frac{\pi}{3}$ and $\cos 9t$ has period $\frac{\pi}{3}$ but has the period of $\sin 3t$; (l) not periodic.

1.35. (a) neither odd nor even; (b) even; (c) odd since $\sin x$ is odd; (d) odd since product of odd and even functions; (e) even; (f) even; (g) neither odd nor even.

1.36. (a) $\frac{1}{\sqrt{x-2}} - \frac{1}{\sqrt{x+3}}$; (b) $-\frac{1}{x+1} + \frac{2}{x+2}$; (c) $\frac{1}{x} + \frac{1}{x-1}$.
Figure 25: Problem 1.32b

Figure 26: Problem 1.32c

(d) \( \frac{1}{x^2} - \frac{1}{x^2 + 1} + \frac{1}{2(x^2 + 1)}; \) (e) \( \frac{1}{x - 1} - \frac{1}{2} + \frac{1}{2(x + 1)}; \)
(f) \( \frac{1}{x + 2} - \frac{1}{2(x^2 + 2)} - \frac{1}{4(x^2 + 2)}; \) (g) \( \frac{1}{x + 1} = \frac{1}{2(x + 1)^2}; \)
(h) \( \frac{1}{2(x - 1)} + \frac{1}{2(x + 1)}; \) (i) \( \frac{1}{2} = \frac{1}{x - 1} + \frac{1}{(x - 1)^2}; \)
(j) \( \frac{1}{x^2} - \frac{1}{x + 1}. \)

1.37. (a) \( \frac{1}{2} = \frac{x + 1}{x^2 + x + 1}; \) (b) \( \frac{1}{2(x - 1)} + \frac{1}{2(x^2 + 1)}; \) (c) \( -\frac{1}{5(x + 1)} + \frac{x + 6}{5(x^2 + 2x + 6)}). \)

1.38. (a) \( \frac{1}{x^2} - \frac{1}{x + 1}; \) (b) \( x - 3 - \frac{1}{x + 1} + \frac{8}{x^2 + 2}; \)
(c) \( 1 + \frac{9}{8(x + 3)} + \frac{1}{8(x + 3)} - \frac{9}{8(x + 3)}). \)

1.39. (a) \( 4 + 8 + 16 + 32; \) (b) \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{17}; \)
(c) \( x + 2x^2 + 3x^3 + 4x^4. \)

1.40. For (a), (b), (c), (e) and (f) proceed as in Example 1.17.
(a) \( 2[1 - (\frac{2}{3})^8] = \frac{255}{128}. \)
(b) \( \frac{1}{2} \cdot \frac{1}{3}[1 - (\frac{1}{12})^5] = \frac{121}{225}. \)
(c) \( (1 - e^{-12})/(1 - e^{-2}). \)
(d) The sum is 642. More generally, let
\[ x + 2x^2 + \cdots + nx^n = T. \]

Then
\[ T - xT = (1 - x)T = xS - nx^{n + 1}, \]
where \( S \) is the sum of the geometric series
\[ 1 + x + \cdots + x^{n - 1}. \]
For the given problem $x = 2$.

(e) $-\frac{3}{8}[1 - (-\frac{1}{2})^{10}] = -\frac{341}{1024}$.

(f) $2[\frac{1-0.9^2}{0.5}] + 3[\frac{1-(0.6)^2}{0.4}] = 11.258 \ldots$

1.41. The series can be expressed as

$$x + x^3 + x^9 + \cdots + x^{41} = x \sum_{n=0}^{10} (x^4)^n.$$

Using (1.33), the sum of the series is

$$\frac{1 - x^{44}}{1 - x^4}.$$

1.42. Let $D$ be the foot of the perpendicular on to the side $AC$. Then

$$c^2 = DB^2 + DA^2 = DB^2 + (AC - DC)^2.$$

But $DB = a \sin C$ and $DC = a \cos C$. Therefore

$$c^2 = a^2 \sin^2 C + (b - a \cos C)^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C = a^2 + b^2 - 2ab \cos C.$$

1.43. The ratio of any pair of successive terms is

$$\frac{f(t_0 + (n+1)T)}{f(t_0 + nT)} = \frac{Ae^{c(t_0+(n+1)T)}}{Ae^{c(t_0+nT)}} = e^{cT},$$
which is independent of $n$. The common ratio is $e^T$.

1.44. (a) $1.111\ldots = 1 + \frac{1}{10} + \frac{1}{100} + \ldots$ is an infinite geometric series with common ratio $\frac{1}{10}$ and sum $1/(1 - \frac{1}{10}) = 11/9$. (b) The common ratio is 1/10, and as a fraction the sum is 1; (c) the common ratio is 1/100, and as a fraction the sum is 1/99; (d) the common ratio is 1/100, and as a fraction the sum is 1/11; (e) the common ratio is 1/10 and as a fraction the sum is 2/3; (f) the notation means $2.72 = 2.727272\ldots$: the common ratio is 1/100 and the sum represents the fraction 30/11.

1.45. The sum of the infinite geometric series is

$$\sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x}, \quad |x| < 1.$$ 

(a) 2; (b) 10/9; (c) $e/(e - 1)$; (d) $\frac{1}{2}$; (e) 3/5.

1.46. (a) 24, 720, 5040; (b) 12; (c) 720; (d) 220; (e) 120; (f) 1, 3, 3, 1.

1.47. (a) (i) $n(n-1)$; (ii) $(n+1)n$; (b) (i) $2^m m!$; (ii) $(2m+1)!/(2^m m!)$.

1.48. (a) (i) 120; (ii) 504; (iii) 120; (iv) 35; (v) 35; (vi) 252; (vii) 4950; (viii) $\binom{10}{4} = 10 C_7 = 120$.

(b) $n P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ Also $n P_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = n!$. Consider a collection of $n$ different letters. The number of different words of length $n$ which can be made without repetition is $n P_n = n!$. Suppose that the letters are $A, B, C, \ldots$. Suppose that the first letter of an $n$ character word is $A$. Then the remaining $n-1$ letters can be chosen in $(n-1)!$ ways. Repeat the procedure for words with first letters $B, C$ and so on. We obtain all words with $n$ characters again, and there are $(n(n-1)! = n!$ of them.

1.49. (a) $4 P_4 = 4!$; (b) $4 C_3 = 4$; (c) $4^4 = 256$; (d) 20; (e) $4 P_4 + 4 P_3 + 4 P_2 + 4 P_1 = 4+12+24+24 = 64$.

(f) Without repetitions the number of combinations is $4 C_1 + 4 C_2 + 4 C_3 + 4 C_4 = 4 + 6 + 4 + 1 = 15$.

With 2 letters the same there are 4 + 12 + 12 = 28 possibilities, and with 3 letters the same there are 4 + 12 = 16. Hence the total number of combinations is 15 + 28 + 16 = 59

1.50. (a) With no E’s there are $4 P_3 = 24$ words, with 1 E there are $3 \times 4 P_2 = 36$, and with 2 E’s there are $3 \times 4 P_1 = 12$. Hence there are 24 + 36 + 12 = 72 words.

(b) Label six letters $A, B, C, D, E_1, E_2$. Then the number of words treating $E_1$ and $E_2$ as distinct is $6! = 720$. The letters $E_1$ and $E_2$ can be interchanged in $2! = 2$ ways. Hence the number of six-letter words is $720/2 = 360$.

1.51. (a) There are $5 P_4 = 5!/1! = 120$ distinct four-digit numbers.

(b) To be divisible by 5, the last digit must be 5. The preceding 3 digits can be chosen in $4 P_3 = 24$ ways. Hence there are 24 numbers divisible by 5.

(c) To be divisible by 2 the final digit must be 2 or 4. As in (b) the number of numbers is $2 \times 4 P_3 = 48$.

(d) The numbers contain either 1, 2, 3 or 4 digits. There are 4 one-digit numbers (excluding zero). For two-digit numbers we must exclude those starting with zero since they are the same as the one-digit numbers. Hence there are 16 distinct two-digit numbers. Similarly there are $4 \times 4 P_2 = 48$ three-digit numbers and $4 \times 4 P_3 = 96$ four-digit numbers. Hence the total number is $4 + 16 + 48 + 96 = 164$ words.

1.52. (a) Without restriction, the number of distinct combinations of personnel (no distinction being made as to which particular post is assigned to each person) is $7 C_4 = 7!/4!3! = 35$.

(b) There is one selection with 4 females, 12 with 3 females and one male, 18 with 2 females and 2 males and 4 with one female and three males. (b) The posts can be filled in the following ways:

- $4 C_1 = 1$ with 4 females;
- $4 C_3 C_1 = 12$ with 3 females and one male;
- $4 C_2 C_2 = 18$ with 2 females and 2 males;
- $4 C_1 C_3 = 12$ with 2 females and 1 male;
- $4 C_2 C_1 = 18$ with 2 males and 2 females;
- $4 C_1 C_2 = 4$ with one female and 3 males.

This confirms the 35 combinations of personnel.
1.53. (a) We may model the problem by thinking of an ordered line of \(N\) pool balls, of various colours (types) denoted by \(A, B, \ldots\), the number of each colour being \(N_A, N_B, \ldots\). The number of possible orders (permutations) for the individual balls is \(N!\), but we cannot distinguish visually between balls having the same colour, so many of the \(N!\) orders will look identical.

Suppose that the number of distinguishable arrangements is \(M\). Each one of these corresponds to a possible \(N_A!N_B!\ldots\) permutations within the separate colours, so that

\[
N! = M[N_A!N_B!\ldots], \quad \text{or} \quad M = \frac{N!}{N_A!N_B!\ldots}.
\]

(b) We require the total number of different combinations, involving every number 1, 2, \ldots, \(N\) of balls. Consider any one of these: it contains 0 or 1 or 2\ldots or \(N_A\) (that is, \((1+N_A)\) possibilities) of type \(A\); 0 or 1 or 2\ldots or \(N_B\) of type \(B\); and so on. The number of possible combinations is therefore

\[
(1+N_A)(1+N_B)\ldots - 1,
\]

in which the term \(-1\) is introduced to exclude the case of an all-zero ‘combination’.

1.54. (a) The national groups may be ordered (permuted) in 4! ways. By allowing for 5! permutations possible within each group we obtain

\[5!5!5!5!4! = 5!^44! = 2880\]

distinct line-ups.

(b) The number of distinct orderings of the 4 groups around a circular table is \((4-1)! = 3!\) (see Example 1.23). All possible permutations within the groups are then to be allowed for, so the total number of arrangements is \(5!3! = 720\).

1.55. (a) (Prizes identical) The number of combinations of 3 distinct prizewinners out of 10 eligibles is \(10C_3 = 120\).

(b) (Prizes different) Call the Prizes \(P_1, P_2, P_3\). \(P_1\) may go to any of 10 people; with each allocation \(P_2\) may go to any of the remaining 9; then \(P_3\) to any of the remaining 8; all of these distributions being distinct. The total number of possibilities is \(10 \times 9 \times 8 = 720\).

(c) (Prizes equal, distribution arbitrary) There are 3 types of distribution which can occur:

(i) One person gets all the prizes: there are 10 possibilities.

(ii) There are 10 persons who might get 2 prizes. With each of these there are 9 persons eligible for the other prize. There are therefore \(9 \times 10 = 90\) possibilities.

(iii) Three different people get prizes. Part (a) gives the number: there are 120 possibilities.

Therefore the total number of possibilities is

\[10 + 90 + 120 = 220\]

(d) \(P_1\) may go to any of the 10; similarly with \(P_2\) and \(P_3\). Therefore the total number is \(10 \times 10 \times 10 = 1000\).

1.56. (a) The table shows the permissible numbers in the 3 categories. The number of combinations possible within each category are given in brackets.

<table>
<thead>
<tr>
<th>Accountants</th>
<th>Lawyers</th>
<th>Doctors</th>
<th>Committees</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>1(2)</td>
<td>3(1)</td>
<td>2</td>
</tr>
<tr>
<td>-</td>
<td>2(1)</td>
<td>2(3C_2)</td>
<td>3</td>
</tr>
<tr>
<td>1(2)</td>
<td>-</td>
<td>3(1)</td>
<td>2</td>
</tr>
<tr>
<td>1(2)</td>
<td>1(2)</td>
<td>2(3C_2)</td>
<td>12</td>
</tr>
<tr>
<td>1(2)</td>
<td>2(1)</td>
<td>1(3)</td>
<td>6</td>
</tr>
<tr>
<td>2(1)</td>
<td>-</td>
<td>2(3C_2)</td>
<td>3</td>
</tr>
<tr>
<td>2(1)</td>
<td>1(2)</td>
<td>1(3)</td>
<td>6</td>
</tr>
<tr>
<td>2(1)</td>
<td>2(1)</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

Check: \(7C_4 = 35\)

12
Committees with exactly 1 accountant: $2 + 12 + 6 = 20$.
Committees with exactly 1 doctor: $6 + 6 = 12$.

(b) To locate the fallacy consider combinations of the 7 letters $A, B, C, D, E, F, G$, and take $n = 4$ and $r = 3$. Take, say, the $r = 3$-fold combination $ABC$ and supplement it by, say, the unused letter $D$, to form the combination $ABCD$. In the fallacious construction this will be counted several times; for example, the same combination is counted again when it arises from supplementing $BCD$ by $A$.

The result is shown to be false by simply substituting the given numbers: only one contradiction is sufficient to dispose of it.

1.57. (a) Refer back to (1.44).
(b) $(1 - x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
(c) 

\[
(x + x^{-1})^5 = x^5 + 5x^4x^{-1} + 10x^3x^{-2} + 10x^2x^{-3} + 5xx^{-4} + x^{-5}
\]

\[= x^5 + 5x^3 + 10x + 10x^{-1} + 5x^{-3} + x^{-5}\]

\[
(x - x^{-1})^5 = x^5 + 5x^4(-x)^{-1} + 10x^3(-x)^{-2} + 10x^2(-x)^{-3} + 5x(-x)^{-4} + (-x)^{-5}
\]

\[= x^5 - 5x^3 - 10x - 10x^{-1} + 5x^{-3} - x^{-5}\]

1.58.

\[(1.01)^{10} = (1 + 0.01)^{10} = 1 + 10 \times (0.01) + 45 \times (0.01)^2 + 120 \times (0.01)^3 + \cdots = 1.01010045 + 0.00012000012 = 1.105\]

to three decimal places.

Similarly

\[(0.99)^8 = (1 - 0.01)^8 = 1 - 8 \times (0.01) + 28 \times (0.01)^2 - 56 \times (0.01)^3 + \cdots = 1 - 0.0800028 - 0.000056 + \cdots = 0.923\]

to 3 decimal places.

1.59. Use the binomial theorem in the form

\[(1 + x)^n = 1 + nC_1x + nC_2x^2 + \cdots + nC_nx^n\]

(a) Put $x = 2$, so that

\[3^n = 1 + 2nC_1 + 2^2nC_2 + \cdots + 2^nC_n\]

For the second result put $x = -1$:

\[0 = 1 - nC_1 - nC_2 - \cdots + (-1)^nC_n\]

(b) Obtain two series with $x = 1$ and $x = -1$. Then add and subtract the series.

1.60. $F(n, k)$ is defined for $n = 0, 1, 2, \ldots$, and $k = 0, 1, 2, \ldots, n$ by

\[F(n, k) = nC_0 + n+1C_1 + n+2C_2 + \cdots + n+kC_k\]

A certain formula, namely

\[F(n, k) = n+k+1C_k\]
is suggested for the sum in (i), and its truth for small values of \( k \) can be confirmed by calculation; for example, from (i)

\[
F(n, 0) = _nC_0 = \frac{n!}{0!n!} = 1 \equiv _{n+1}C_0, \quad (iii)
\]

\[
F(n, 1) = _nC_0 + _{n+1}C_1 = 1 + \frac{(n + 1)!}{1!n!} = n + 2 \equiv _{n+2}C_1;
\]

and so on.

To prove the truth of (ii) for all values of \( 0 \leq k \leq n \), recast (i) into the form

\[
F(n, k + 1) \equiv F(n, k) + _{n+k+1}C_{k+1}, \quad (iv)
\]

a ‘recurrence formula’ enabling us to advance one step at a time in \( k \), starting, for example, with \( F(n, 0) \) and finding \( F(n, 1) \), \( F(n, 2) \), . . . , successively.

Now suppose we have verified the formula (ii) for any one particular value of \( k \), say for \( k = K \);

that is, we know somehow that

\[
F(n, K) = _{n+K}C_K = \frac{(n + K + 1)!}{K!(n + 1)!} \quad (v)
\]

(for all \( n \)). Then from (iv)

\[
F(n, K + 1) = F(n, K) + _{n+K+1}C_{K+1}
\]

\[
= \frac{(n + K + 1)!}{K!(n + 1)!} + \frac{(n + K + 1)!}{(K + 1)!n!}
\]

\[
= \frac{(n + K + 1)!}{K!n!} \left( \frac{1}{n + 1} + \frac{1}{K + 1} \right)
\]

\[
= \frac{(n + K + 2)!}{(K + 1)!(n + 1)!} = _{n+K+2}C_{K+1}
\]

We have proved that if (iv) is true for \( k = K \), it is true for \( k = K + 1 \), where \( K \) may take any value in \( 0 \leq K < n \).

But we verified in (iii) that (iv) holds good when \( k = K = 0 \). Therefore by (vi) it is true when \( k = K + 1 = 1 \), so by (vi) again it is true when \( k = K + 2 = 2 \), and so on. It is therefore true for all \( k \).

1.61. Using partial fractions

\[
\frac{1}{x^2 + 3x + 2} = \frac{1}{(1 + x)(2 + x)} = \frac{1}{1 + x} - \frac{1}{2 + x}.
\]

Write as

\[
\frac{1}{x^2 + 3x + 2} = (1 + x)^{-1} - \frac{1}{2}(1 + \frac{1}{2}x)^{-1},
\]

and expand both terms using (1.37) for infinite geometric series. Hence

\[
\frac{1}{x^2 + 3x + 2} = (1 - x + x^2 - x^3 + \cdots) - \frac{1}{2}(1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} + \cdots)
\]

\[
= \frac{1}{2} - (1 - \frac{1}{2.2})x + (1 - \frac{1}{2.2.4})x^2 - (1 - \frac{1}{2.2.4.6})x^3 + \cdots.
\]

1.62. \( V_1 = A(1 + R) \), \( V_2 = V_1(1 + R) = A(1 + R)^2 \), etc. In pounds: for 1000 @ 3% p.a.;

\[
V_5 = 1000(1 + 0.03)^5 = 1159.27, \quad V_{10} = 1343.92; \quad V_{15} = 1557.97.
\]
(b) Let the period start at an arbitrary time \( T_0 \). Then

\[
\frac{V_{T_0 + T}}{V_{T_0}} = \frac{(1 + R)^{T_0 + T}}{(1 + R)^{T_0}} = (1 + R)^T.
\]

(c) Let \( T_2 \) be the doubling period, so that from (b)

\[
(1 + R)^{T_2} = 2 \quad \text{and} \quad T_2 = \frac{\ln 2}{\ln(1 + R)}.
\]

If \( R = 3\% \), \( T_2 = 23.4 \) yr; if \( R = 6\% \), \( T_2 = 11.9 \) yr; if \( R = 9\% \), \( T_2 = 8.0 \) yr.

For the ten-times period, \( T_{10} = \frac{\ln 10}{\ln(1 + R)} \). If \( R = 6\% \), then \( T_{10} = 39.6 \) yr.

1.63. If the income is withdrawn annually, it has been allowed to accrue to the fund through the previous year with interest at the going rate \( R \) annually, or \( r \) monthly, the relation being

\[
A(1 + R) = A(1 + r)^{12}
\]

where \( A \) is the value of the fund at the start of that year. By the binomial theorem,

\[
(1 + r)^{12} = 1 + 12r + \cdots + 1 + 12r,
\]

so \( R > 12r \).

1.64. If the interest is payable monthly at the rate of \( r_M \) per month, the interest on a fixed debt \( D \) over any 12-month period is \( D(1 + r_M)^{12} \). This is equal to \( D(1 + R) \) where \( R \) is the annual equivalent rate (AER). Therefore \( R = (1 + r_M)^{12} - 1 \). If \( r_M = 1\% \), \( R = 1.01^{12} - 1 = 0.126 \) (12.6\%). If \( r_M = 3\% \), \( R = 0.425 \) (42.5\%).

1.65. (a) After \( N \) complete years the initial payment \( A \) has drawn interest for \( N \) yrs, the second payment for \( N - 1 \) yrs, and so on, and the \((N - 1)\)th payment for 1 yr. The value \( V_N \) of the fund is then given by the geometric series

\[
A(1 + R)^N + A(1 + R)^{N-1} + \cdots + A(1 + R) = A(1 + R)\{1+(1+R)+\cdots+(1+R)^{N-1}\}
\]

\[
= A(1 + R)(1 + (1 + R) + \cdots + (1 + R)^{N-1})/R.
\]

(b) \( N = 10 \), \( R = 5\% \). We obtain

\[
V_{10} = \frac{100(1.05)(1.05^{10} - 1)}{0.05} = £1320.68,
\]

equivalent to a gain of 32\% on the total investment of £1000.

(c) \( M \) investments of \( 2A \), at 2-year intervals. Formula (a), with the fund value increasing by a factor \((1 + R)^2\) in each interval, becomes

\[
V_{2M} = \frac{(2A)(1 + R)^2\{(1 + R)^2 - 1\}}{(1 + R)^2 - 1}.
\]

Using the data in (b) we obtain

\[
V_{10} = \frac{200(1.05)^2(1.05^{10} - 1)}{1.05^2 - 1} = £1352.88.
\]

Chapter 2: Differentiation

2.1. Below are some sample values for three values of \( x \) on either side of the point where the tangent is required. (The exact value of the slope is also given here.)
(a) \( y = x^3 \) at (1, 1).

\[
\begin{array}{c|cccccc}
 x & 0.94 & 0.96 & 0.98 & 1.02 & 1.04 & 1.06 \\
\hline
\text{chord slope} & 2.82 & 2.88 & 2.94 & 3.06 & 3.12 & 3.18 \\
\end{array}
\]

The slope is 3.

(b) \( y = \sqrt{x} \) at (1, 1).

\[
\begin{array}{c|cccccc}
 x & 0.85 & 0.90 & 0.95 & 1.025 & 1.10 & 1.15 \\
\hline
\text{chord slope} & 0.520 & 0.513 & 0.506 & 0.494 & 0.488 & 0.483 \\
\end{array}
\]

The slope is 0.5.

(c) \( y = \cos x \) at \((\frac{1}{2}\pi, 1/\sqrt{2})\).

\[
\begin{array}{c|cccccc}
 x - \frac{1}{2}\pi & -0.09 & -0.06 & -0.03 & +0.03 & 0.06 & 0.09 \\
\hline
\text{chord slope} & 0.674 & 0.685 & 0.696 & 0.718 & 0.728 & 0.738 \\
\end{array}
\]

The slope is \(1/\sqrt{2} = 0.707\).

(d) \( y = e^x \) at (0, 1).

\[
\begin{array}{c|cccccc}
 x & -0.15 & -0.10 & -0.05 & 0.05 & 0.10 & 0.15 \\
\hline
\text{chord slope} & 0.929 & 0.952 & 0.975 & 1.025 & 1.052 & 1.079 \\
\end{array}
\]

The slope is 1.

(e) \( y = e^{2x} \) at (0, 1).

\[
\begin{array}{c|cccccc}
 x & -0.15 & -0.10 & -0.05 & 0.05 & 0.10 & 0.15 \\
\hline
\text{chord slope} & 1.728 & 1.813 & 1.903 & 2.103 & 2.214 & 2.332 \\
\end{array}
\]

The slope is 2.

(f) \( y = x^3 + x^{\frac{1}{2}} \) at (1, 2).

\[
\begin{array}{c|cccccc}
 x & 0.94 & 0.96 & 0.98 & 1.02 & 1.04 & 1.06 \\
\hline
\text{chord slope} & 3.33 & 3.38 & 3.44 & 3.56 & 3.62 & 3.68 \\
\end{array}
\]

The slope is 3.5, the sum of the slopes in (a) and (b).

(g) \( y = \ln x \) at (1, 0).

\[
\begin{array}{c|cccccc}
 x & 0.94 & 0.96 & 0.98 & 1.02 & 1.04 & 1.06 \\
\hline
\text{chord slope} & 1.031 & 1.0206 & 1.010 & 0.990 & 0.981 & 0.971 \\
\end{array}
\]

The slope is 1.

2.2. (a) For \( y = 3x \) at (2, 6),

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{3(2 + \delta x) - 6}{\delta x} = 3.
\]

(b) For \( y = 3 - 2x \) at (1, 1),

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{3 - 2(1 + \delta x) - (3 - 2)}{\delta x} = -2.
\]

(c) For \( y = 3x^2 \) at (1, 3),

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \left[ \frac{3(1 + \delta x)^2 - 3(1)^2}{\delta x} \right] = \lim_{\delta x \to 0} [6 + 3\delta x] = 6.
\]

(d) For \( y = x^3 \) at (1, 1),

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \left[ \frac{(1 + \delta x)^3 - 1^3}{\delta x} \right] \\
= \lim_{\delta x \to 0} \left[ \frac{3\delta x + 3(\delta x)^2 + (\delta x)^3}{\delta x} \right] \\
= \lim_{\delta x \to 0} [3 + 3\delta x + (\delta x)^2] = 3.
\]
(e) For \( y = \frac{1}{x} \) at \((2, \frac{1}{2})\),
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[ \frac{1}{2 + \delta x} - \frac{1}{2} \right] = \lim_{\delta x \to 0} \left[ \frac{-1}{2(2 + \delta x)} \right] = -\frac{1}{4}.
\]

(f) For \( y = 3x + 2x^2 \) at \((1, 5)\),
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[ \frac{3(1 + \delta x) + 2(1 + \delta x)^2 - 3 - 2}{\delta x} \right] = \lim_{\delta x \to 0} \frac{1}{\delta x} [3 + 2\delta x] = 7.
\]

(g) For \( y = (1 + 2x)^2 = 1 + 4x + 4x^2 \) at \((-1, 1)\),
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[ 1 + 4(-1 + \delta x) + 4(-1 + \delta x)^2 - 1 + 4 - 4 \right] = \lim_{\delta x \to 0} \frac{1}{\delta x} [4 + 4(-2 + \delta x)^2] = -4.
\]

2.3. (a) \( y = 3x^2 \)
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} [3(x + \delta x)^2 - 3x^2] = \lim_{\delta x \to 0} \frac{1}{\delta x} [6x + 3\delta x] = 6x.
\]

(b) \( y = x^3 \).
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} [(x + \delta x)^3 - x^3] = \lim_{\delta x \to 0} \frac{1}{\delta x} [3x^2 + 3x\delta x + (\delta x)^2] = 3x^2.
\]

(c) \( y = \frac{1}{x} \).
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[ \frac{1}{x + \delta x} - \frac{1}{x} \right] = \lim_{\delta x \to 0} \left[ \frac{-1}{x(x + \delta x)} \right] = -\frac{1}{x^2}.
\]

(d) \( y = x + \frac{1}{2} \).
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} [(x + \delta x + \frac{1}{2}) - (x + \frac{1}{2})] = 1.
\]

(e) \( y = x + \frac{1}{x} \).
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[ \left( x + \delta x + \frac{1}{x + \delta x} \right) - \left( x + \frac{1}{x} \right) \right] = \lim_{\delta x \to 0} \left[ 1 - \frac{1}{x^2 + x\delta x} \right] = 1 - \frac{1}{x^2}.
\]
(f) \( y = 2x^2 - 3. \)

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x} [(2(x + \delta x)^2 - 3) - (2x^2 - 3)]
= \lim_{\delta x \to 0} [4x + 2\delta x]
= 4x.
\]

2.4. Let \( x = f(t) \) be the displacement function in each case. The average velocity over the interval \( t \) to \( t + \delta t \) equals

\[
\frac{[f(t + \delta t) - f(t)]}{\delta t}.
\]

(a) \( x = f(t) = 3t. \) When \( t = 1. \)

<table>
<thead>
<tr>
<th>Interval ( \delta t )</th>
<th>0.5</th>
<th>0.25</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(1 + \delta t) )</td>
<td>4.5</td>
<td>3.75</td>
<td>3.3</td>
<td>3.03</td>
</tr>
<tr>
<td>Average velocity</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

(Since \( f(t) \) is linear in \( t \) the velocity 3 at all \( t \).)

(b) \( x = f(t) = 5t^2. \) When \( t = 3. \)

<table>
<thead>
<tr>
<th>Interval ( \delta t )</th>
<th>0.5</th>
<th>0.25</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(3 + \delta t) )</td>
<td>61.25</td>
<td>52.81</td>
<td>48.05</td>
<td>45.35</td>
</tr>
<tr>
<td>Average velocity</td>
<td>32.5</td>
<td>31.25</td>
<td>30.5</td>
<td>30.5</td>
</tr>
</tbody>
</table>

The values are approaching the limit 30.

(c) \( x = f(t) = 2t - 5t^2. \) When \( t = 1. \)

<table>
<thead>
<tr>
<th>Interval ( \delta t )</th>
<th>0.5</th>
<th>0.25</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(1 + \delta t) )</td>
<td>−8.25</td>
<td>3.75</td>
<td>−3.85</td>
<td>−3.08</td>
</tr>
<tr>
<td>Average velocity</td>
<td>−10.5</td>
<td>−9.25</td>
<td>−8.5</td>
<td>−8.05</td>
</tr>
</tbody>
</table>

The limit is −8.

(d) \( x = 2t - 5t^2. \) When \( t = 0.2. \)

<table>
<thead>
<tr>
<th>Interval ( \delta t )</th>
<th>0.5</th>
<th>0.25</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(0.2 + \delta t) )</td>
<td>−1.25</td>
<td>−0.3125</td>
<td>−0.05</td>
<td>−0.0005</td>
</tr>
<tr>
<td>Average velocity</td>
<td>−25</td>
<td>−1.25</td>
<td>−0.5</td>
<td>−0.05</td>
</tr>
</tbody>
</table>

In the limit the velocity is zero.

2.5. (a) \( \frac{dy}{dx} = 1 \) for all \( x; \) (b) \( \frac{dy}{dx} = 3x^2 \) so that \( \frac{dy}{dx} = 27 \) at \( x = 3; \)
(c) \( \frac{dy}{dx} = 4x^3 \) so that \( \frac{dy}{dx} = 32 \) at \( x = 2 \) and −32 at \( x = −2. \)

2.6. (a) \( y = x; \) \( \frac{dy}{dx} = 1: \) the graph is a straight line at 45° to the \( x \) axis.

(b) \( y = x^2; \) \( \frac{dy}{dx} = 2x: \) the slope is negative for \( x < 0 \) and positive for \( x > 0, \) and increases from \( −\infty \) to \( +\infty: \) the curve is a parabola.
(c) $y = x^3; \frac{dy}{dx} = 3x^2$: the slope is positive except at $x = 0$ where it is zero.
(d) $y = x^4; \frac{dy}{dx} = 4x^3$.

(e) $y = x^5; \frac{dy}{dx} = 5x^4$.

2.7. For the displacement $x = t^3$, the velocity of the point is $\frac{dx}{dt} = 3t^2$ and its acceleration is $\frac{d^2x}{dt^2} = 6t$. The graph of acceleration against time is a straight line.

2.8. (a) If $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dr} = 4\pi r^2$.
(b) If $S = \pi d^2$ then $\frac{dS}{dd} = 2\pi d$.
(c) If $E = kT^4$ then $\frac{dE}{dt} = 4kT^3$.
(d) If $I = V/R$ then $\frac{dI}{dV} = 1/R$.
(e) If $H = RI^2$ then $\frac{dH}{dI} = 2RI$.
(f) If $V = RT/P$ then $\frac{dV}{dT} = R/P$.

2.9.
(a) \[
\frac{d}{dx}(3x^2 - 2x + 1) = 3\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(1) = 6x - 2.
\]
(b) \[
\frac{d}{dx}(x^7 - 3x^6 + x + 1) = 7x^6 - 18x^5 + 1.
\]

(c) \[
\frac{d}{dx}(x + C) = 1.
\]

(d) \[
\frac{d}{dx}[x(x - 1)] = \frac{d}{dx}(x^2 - x) = 2x - 1.
\]

(e) \[
\frac{d}{dx}[x^2(x^2 + 1)] = \frac{d}{dx}[x^4 + x^2 + 1] = 4x^3 + 2x.
\]

(f) \[
\frac{d}{dx}(ax^2 + bx + c) = 2ax + b.
\]

(g) \[
\frac{d}{dx}[(x - 1)^2] = \frac{d}{dx}(x^2 - 2x + 1) = 2x - 2.
\]

2.10. Let \(m_1\) and \(m_2\) be the slopes of the curves at the point of intersection, and check that \(m_1m_2 = -1\). Then
(a) \(m_1 = (d/dx)(1 + x - x^2) = 1 - 2x = -1 \text{ at } x = 1,
\)
\(m_2 = (d/dx)(1 - x + x^2) = -1 + 2x = 1 \text{ at } x = 1. \) Hence \(m_1m_2 = -1\) as required.
(b) \(m_1 = -x = -1, m_2 = 1 \text{ at } x = 1.
\)
(c) \(m_1 = -x = -1, m_2 = x = 1 \text{ at } x = 1.
\)

2.11. (a) The curves \(y = x^2\) and \(y = 1 - x^2\) intersect where \(x^2 = 1 - x^2\) or where \(x^2 = 1/2\). Hence the points of intersection occur at \(A : (\sqrt{2}/2, 1/2)\) and \(B : (-\sqrt{2}/2, 1/2)\)

The slopes of the curves at \(A\) are
\(m_1 = 2x = 2/\sqrt{2} = \sqrt{2}\) and \(m_2 = -2x = -2/\sqrt{2} = -\sqrt{2}\).

Let
\(\tan \alpha_1 = \sqrt{2} \quad (0 < \alpha_1 < \pi/2)\) and \(\tan \alpha_2 = -\sqrt{2} \quad (-\pi/2 < \alpha_1 < 0)\).

Using the identity from (1.17a):
\[
\tan(\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2} = \frac{\sqrt{2} + \sqrt{2}}{1 - 2\sqrt{2}} = -2\sqrt{2}
\]

We choose a positive value for the angle (a sketch of the intersection of the curves is helpful).

Hence \(\alpha_1 - \alpha_2 = \arctan(-2\sqrt{2}) = 109.47^\circ\).

The slopes of the curves at \(B\) are
\(n_1 = -2x = -2/\sqrt{2} = -\sqrt{2}\) and \(n_2 = 2x = 2/\sqrt{2} = \sqrt{2}\).

The two slopes at \(B\) are interchanged but otherwise the same. Hence the angle between the tangents will also be 109.47°. (Note that in both these cases you might obtain the alternative angles \((180 - 109.47)^\circ\).)

(b) The curves \(y = \frac{1}{3}x^3\) and \(y = x^2 - 2x + \frac{4}{3}\) intersect where
\(x^3 = 3x^2 - 6x + 4\) or where \((x - 1)(x^2 - 2x + 4) = 0\).

The only real root is \(x = 1\). Hence the point of intersection is at \((1, \frac{1}{3})\). The slopes of the curves at this point are \(m_1 = 1\) and \(m_2 = 0\). Let \(\tan \alpha_1 = 1\) and \(\tan \alpha_2 = 0\). Then we can choose \(\alpha_1 = \frac{\pi}{4}\) and \(\alpha_2 = 0\). The required angle is \(\frac{\pi}{4}\).
2.12. Use the limits in Section 2.6. Note that, in all the following, \( \varepsilon \) is never zero, so cancellation is legitimate.

(a) \[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} 1 = 1.
\]

(b) \[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{2} = \frac{1}{2}.
\]

(c) \[
\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0} \varepsilon = 0.
\]

(d) \[
\lim_{\varepsilon \to 0} \frac{e^{2\varepsilon} - 1}{2\varepsilon} = \lim_{\mu \to 0} \frac{e^{\mu} - 1}{\mu} = 1, \quad (\mu = 2\varepsilon) \text{ (from (2.11))}.
\]

(e) \[
\lim_{\varepsilon \to 0} \frac{e^{2\varepsilon} - 1}{\varepsilon} = \lim_{\mu \to 0} 2 \frac{e^{\mu} - 1}{\mu} = 2, \quad (\mu = 2\varepsilon).
\]

(f) \[
\lim_{\varepsilon \to 0} \frac{\sin 2\varepsilon}{2\varepsilon} = \lim_{\mu \to 0} \frac{\sin \mu}{\mu} = 1, \quad (\mu = 2\varepsilon) \text{ (from (2.13)).}
\]

(g) \[
\lim_{\varepsilon \to 0} \frac{\sin 2\varepsilon}{\varepsilon} = \lim_{\mu \to 0} 2 \frac{\sin \mu}{\mu} = 2, \quad (\mu = 2\varepsilon).
\]

(h) \[
\lim_{\varepsilon \to 0} \frac{\ln(1 + \varepsilon^2)}{\varepsilon^2} = \lim_{\mu \to 0} 2 \frac{\ln(1 + \mu)}{\mu} = 1, \quad (\mu = \varepsilon^2) \text{ (from (2.14)).}
\]

(i) Note that (2.13) is only true if \( \varepsilon \) is measured in radians. Therefore replace \( \varepsilon \) degrees by \( \frac{180\varepsilon}{\pi} \) radians. Hence \[
\lim_{\varepsilon \to 0} \frac{\sin \varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\sin(\pi\varepsilon/180)}{\pi\varepsilon/180} = \lim_{\mu \to 0} \frac{\pi \sin \mu}{180\mu} = \frac{\pi}{180} \text{ (from (2.13)).}
\]

(j) \[
\lim_{\varepsilon \to 0} \tan \varepsilon = \lim_{\varepsilon \to 0} \frac{\sin \varepsilon}{\varepsilon \cos \varepsilon} = \lim_{\varepsilon \to 0} \frac{\sin \varepsilon}{\varepsilon} \lim_{\varepsilon \to 0} \frac{1}{\cos \varepsilon} = 1 \times 1 = 1
\]

(k) \[
\lim_{\varepsilon \to 0} \frac{\sinh \varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{e^\varepsilon - e^{-\varepsilon}}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{e^{2\varepsilon} - 1}{2\varepsilon} \lim_{\varepsilon \to 0} e^{-\varepsilon} = 1 \times 1 = 1
\]

(l) \[
\lim_{\varepsilon \to 0} \frac{e^{-\varepsilon} - 1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{e^\varepsilon - 1}{\varepsilon} \lim_{\varepsilon \to 0} \left[-e^{-\varepsilon}\right] = -1.
\]

2.13.

\[
\frac{d}{dx} (\cos x) = \lim_{\delta x \to 0} \left[ \frac{\cos(x + \delta x) - \cos x}{\delta x} \right] \\
= \lim_{\delta x \to 0} \left[ \frac{-2 \sin \left(\frac{1}{2} (2x + \delta x) \sin \left(\frac{1}{2} (\delta x) \right) \right)}{\delta x} \right] \\
= \lim_{\delta x \to 0} \left[ \frac{- \sin(x + \frac{1}{2} \delta x) \sin \left(\frac{1}{2} \delta x \right)}{\frac{1}{2} \delta x} \right] \\
= - \sin x
\]
2.14. 

(a) \[ \frac{d}{dx}(e^{2x}) = \lim_{\varepsilon \to 0} \left[ \frac{e^{2(x + \varepsilon)} - e^{2x}}{\varepsilon} \right] = 2e^{2x} \lim_{\varepsilon \to 0} \left[ \frac{e^{2\varepsilon} - 1}{2\varepsilon} \right] = 2e^{2x}. \]

(from (2.11)).

(b) \[ \frac{d}{dx}(\sin 2x) = \lim_{\varepsilon \to 0} \left[ \frac{\sin[2(x + \varepsilon)] - \sin 2x}{\varepsilon} \right] = \lim_{\varepsilon \to 0} \left[ \frac{2\sin \varepsilon}{\varepsilon} \cos \frac{1}{2}(4x + 2\varepsilon) \right] = 2 \cos 2x. \]

(c) \[ \frac{d}{dx}(e^{-x}) = \lim_{\varepsilon \to 0} \left[ \frac{e^{-(x + \varepsilon)} - e^{-x}}{\varepsilon} \right] = -e^{-x} \lim_{\varepsilon \to 0} \left[ \frac{e^{-\varepsilon} - 1}{-\varepsilon} \right] = -e^{-x}. \]

Thus \[ \frac{d}{dx} \left( \sinh x \right) = \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2} (e^x + e^{-x}) = \cosh x. \]

\[ \frac{d}{dx} \left( \cosh x \right) = \frac{1}{2} \frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2} (e^x - e^{-x}) = \sinh x. \]

2.15. 

(a) \[ \frac{d}{dx}(2\sin x - 3\cos x) = 2 \frac{d}{dx}(\sin x) - 3 \frac{d}{dx}(\sin x) = 2 \cos x + 3 \sin x. \]

(b) \[ \frac{d}{dx}(\ln 3x) = \frac{d}{dx}(\ln 3 + \ln x) = \frac{1}{x}. \]

(c) \[ \frac{d}{dx} \left( \ln x^3 \right) = \frac{d}{dx} (3 \ln x) = \frac{3}{x}. \]

(d) \[ \frac{d}{dx} (\sin x - x) = \cos x - 1. \]

(e) \[ \frac{d}{dx} \left( e^x - 1 - x - \frac{1}{2} x^2 \right) = e^x - 1 - x. \]

2.16. The required tangent lines are
(a) \( y = 3x - 2; \) (b) \( y = 24x - 39; \) (c) \( y = -x + \frac{1}{2} \pi; \) (d) \( y = x/e; \) (e) \( y = 1; \) (f) \( y = -x + 3. \)

2.17. 

(a) \( y = x^6, \quad \frac{dy}{dx} = 6x^5, \quad \frac{d^2y}{dx^2} = 30x^4, \quad \frac{d^3y}{dx^3} = 120x^3. \)

(b) \( y = 3x^2 - 2x + 2, \quad \frac{dy}{dx} = 6x - 2, \quad \frac{d^2y}{dx^2} = 6, \quad \frac{d^3y}{dx^3} = 0. \)

(c) \( y = x^6 - x^2, \quad \frac{dy}{dx} = 6x^5 - 2x, \quad \frac{d^2y}{dx^2} = 30x^4 - 2, \quad \frac{d^3y}{dx^3} = 120x^3. \)
y = 2 \sin x - 3 \cos x, \quad \frac{dy}{dx} = 2 \cos x + 3 \sin x,

\frac{d^2y}{dx^2} = -2 \sin x + 3 \cos x, \quad \frac{d^3y}{dx^3} = -2 \cos x - 3 \sin x.

y = e^x - 1 - x - \frac{1}{2} x^2, \quad \frac{dy}{dx} = e^x - 1 - x, \quad \frac{d^2y}{dx^2} = e^x - 1, \quad \frac{d^3y}{dx^3} = e^x.

2.18. To prove that \((dx^N/dx^N)(x^N) = N!\) for all integers \(N \geq 1\). We can confirm the formula for the case \(N = 1\):

\[
\frac{d}{dx}(x) = 1 = 1!
\]

as a starting-point in a step-by-step argument.

Suppose that we have somehow established that the result is true for any one particular value of \(N\), say for \(N = K\), so that

\[
\frac{d^N(x^N)}{dx^N} = K! \text{ when } N = K.
\] (ii)

Next, consider the transition to \(N = K + 1\):

\[
\frac{d^{K+1}}{dx^{K+1}}(x^{K+1}) = \frac{d}{dx} \left[ \frac{d^K}{dx^K}(x^{K+1}) \right] = \frac{d^K}{dx^K} \left[ \frac{d}{dx}(x^{K+1}) \right]
\]

\[
= \frac{d^K}{dx^K} \{(K+1)x^K\} \text{ (by (2.9))}
\]

\[
= (K+1)K! \text{ (by (ii))}
\]

\[
= (K+1)!
\]

In other words, if (ii) is true for some integer \(N = K\), it follows that it is also true for \(N = K + 1\). Since we now know it is true for \(N = K + 1\), the same argument implies that it is true for \(N = (K + 1) + 1 = K + 2\); and so on for all subsequent values of \(N\).

But we have verified its truth in the case \(N = 1\) (in equation (i)); therefore (ii) is true for all values of \(N\). This is proof by induction.

2.19. If \(y = x^2(x^2 - 1)\), then

\[
\frac{dy}{dx} = 4x^3 - 6x, \quad \frac{d^2y}{dx^2} = 12x^2 - 6.
\]

(a) The slope of the curve is positive where \(dy/dx > 0\) or where \(x(2x + \sqrt{3})(2x - \sqrt{3}) > 0\). This occurs where \(-\frac{1}{2}\sqrt{3} < x < 0\) and \(\frac{1}{2}\sqrt{3} < x < \frac{1}{2}\sqrt{3}\).

(b) The slope of the curve is negative where \(x(2x + \sqrt{3})(2x - \sqrt{3}) < 0\), that is where \(x < -\frac{1}{2}\sqrt{3}\) and \(0 < x < \frac{1}{2}\sqrt{3}\).

(c) The second derivative is positive where \(12(x + \frac{1}{2}\sqrt{2})(x - \frac{1}{2}\sqrt{2}) > 0\), that is where \(x > \frac{1}{2}\sqrt{2}\) and \(x < -\frac{1}{2}\sqrt{2}\).

(d) The second derivative is negative where \(-\frac{1}{2}\sqrt{2} < x < \frac{1}{2}\sqrt{2}\).

2.20. At \(x = x_0\) the tangent has slope \(m_0 = 2x_0\). Hence the slope of the normal is \(-1/m_0 = -1/(2ax_0)\). The equation of the normal is therefore

\[
y - ax_0^2 = -\frac{1}{2ax_0}(x - x_0).
\]

Chapter 3: Further techniques for differentiation
3.1.

(a) \[
\frac{d}{dx}(xe^x) = x \frac{d}{dx}(e^x) + \frac{d}{dx}(x)e^x = xe^x + e^x.
\]

(b) \[
\frac{d}{dx}(x \sin x) = x \cos x + \sin x.
\]

(c) \[
\frac{d}{dx}(x \cos x) = -x \sin x + \cos x.
\]

(d) \[
\frac{d}{dx}(e^x \sin x) = e^x \cos x + e^x \sin x.
\]

(e) \[
\frac{d}{dx}(x \ln x) = 1 + \ln x.
\]

(f) \[
\frac{d}{dx}(x^2 \ln x) = x^2 \ln x + 2x \ln x = x + 2 \ln x.
\]

(g) \[
\frac{d}{dx}(x^2 \ln x) = \frac{e^x}{x} + e^x \ln x.
\]

(h) \[
\frac{d}{dx}(x^2 e^x) = x^2 e^x + 2xe^x.
\]

(i) \[
\frac{d}{dx}(\sin x \cos x) = -\sin^2 x + \cos^2 x = \cos 2x.
\]

(j) \[
\frac{d}{dx}(x^2 x^3) = x^2 \cdot 3x^2 + 2x \cdot x^3 = 5x^4 = \frac{d}{dx}(x^5).
\]

3.2. All these problems illustrate the reciprocal and quotient rules given in (3.2).

(a) \[
\frac{d}{dx}(\cot x) = \frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x(x \cos x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.
\]

(b) \[
\frac{d}{dx} \left( \frac{x}{x+1} \right) = \frac{1}{(x+1)^2} \left( (x+1) \frac{d}{dx}(x) - x \frac{d}{dx}(x+1) \right) = \frac{1}{(x+1)^2}.
\]

(c) \[
\frac{d}{dx} \left( \frac{\sin x}{x} \right) = \frac{1}{x^2} (x \cos x - \sin x).
\]

(d) \[
\frac{d}{dx} \left( \frac{e^x}{x} \right) = \frac{e^x}{x^2} (x-1).
\]

(e) \[
\frac{d}{dx} \left( \frac{x^2 - 1}{x^2 + 1} \right) = \frac{1}{(x^2+1)^2} [(x^2 + 1)(2x) - (x^2 - 1)(2x)] = \frac{4x}{(x^2+1)^2}.
\]
(f) \[
\frac{d}{dx} \left( \frac{\tan x}{x^2} \right) = \frac{1}{x^4} (x^2 \sec^2 x - 2x \tan x) = \frac{1}{x^3} (x \sec^2 x - 2 \tan x).
\]

(g) \[
\begin{align*}
\frac{d}{dx} \left( \frac{\sin x + \cos x}{\sin x - \cos x} \right) &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
&= -\frac{2}{(\sin x - \cos x)^2}.
\end{align*}
\]

(h) \[
\frac{d}{dx} (\sec x) = \frac{d}{dx} \left( \frac{1}{\cos x} \right) = \frac{\sin x}{\cos^2 x} = \sec x \tan x.
\]

(i) \[
\frac{d}{dx} (\cosec x) = \frac{d}{dx} \left( \frac{1}{\sin x} \right) = -\frac{\cos x}{\sin^2 x} = -\cot x \cosec x.
\]

(j) \[
\frac{d}{dx} \left( \frac{x}{3x^2 - 2} \right) = \frac{-2 - 3x^2}{(-2 + 3x^2)^2}.
\]

(k) \[
\frac{d}{dx} \left( \frac{1}{x(x^3 + 1)} \right) = -\frac{1 - 4x^3}{x^2(x^3 + 1)}.
\]

(l) \[
\frac{d}{dx} \left( \frac{1}{\ln x} \right) = -\frac{1}{x(\ln x)^2}.
\]

(m) \[
\frac{d}{dx} (x^n) = \frac{d}{dx} \left( \frac{1}{x^{-n}} \right) = nx^{n-1} \text{ (by the quotient rule)}.
\]

(n) \[
\frac{d}{dx} \left( \frac{1}{x + 1} \right) = -\frac{1}{(x + 1)^2}.
\]

(o) \[
\frac{d}{dx} (e^{-x}) = \frac{d}{dx} \left( \frac{1}{e^x} \right) = -\frac{1}{e^{2x}} \frac{d}{dx} (e^x) = -e^{-x}.
\]

(p) \[
\frac{d}{dx} \left( \frac{1}{\tan x} \right) = \frac{d}{dx} (\cot x) = -\cosec^2 x \text{ (as in (a))}.
\]

3.3. \[
\frac{d}{dx} \left[ \frac{1}{1 - x} \right] = \frac{1}{(1 - x)^2}; \quad \frac{d^2}{dx^2} \left[ \frac{1}{1 - x} \right] = \frac{2}{(1 - x)^3}; \quad \frac{d^3}{dx^3} \left[ \frac{1}{1 - x} \right] = \frac{6}{(1 - x)^4}.
\]
(b) \[ \frac{d}{dx}(x \sin x) = x \cos x + \sin x; \quad \frac{d^2}{dx^2}(x \sin x) = 2 \cos x - x \sin x; \]
\[ \frac{d^3}{dx^3}(x \sin x) = -x \cos x - 3 \sin x. \]

(c) \[ \frac{d}{dx} \left( \frac{x}{x-1} \right) = -\frac{1}{(x-1)^2}; \quad \frac{d^2}{dx^2} \left( \frac{x}{x-1} \right) = \frac{2}{(x-1)^3}; \]
\[ \frac{d^3}{dx^3} \left( \frac{x}{x-1} \right) = -\frac{6}{(x-1)^4}. \]

(d) Let \( y = f(x)g(x) \). Then
\[ \frac{dy}{dx} = g \frac{df}{dx} + f \frac{dg}{dx}, \]
\[ \frac{d^2y}{dx^2} = 2 \frac{df}{dx} \frac{dg}{dx} + \frac{d^2f}{dx^2} + f \frac{d^2g}{dx^2}, \]
\[ \frac{d^3y}{dx^3} = 3 \frac{dg}{dx} \frac{d^2f}{dx^2} + 3 \frac{df}{dx} \frac{d^2g}{dx^2} + \frac{d^3f}{dx^3} + f \frac{d^3g}{dx^3}. \]

3.4. These problems use the chain rule (3.3) in the form
\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}. \]

(a) Let \( u = \sin x \). Then \( y = u^2 \), and
\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 2u \cos x = 2 \sin x \cos x = \sin 2x. \]

(b) Let \( u = \cos x, y = u^2 \). Then \( (d/\!dx)(\cos^2 x) = -2 \sin x \cos x = -2 \sin 2x. \)

(c) Let \( u = x^2, y = \sin u \). Then \( (d/\!dx)(\sin x^2) = 2x \cos x^2 \).

(d) Let \( u = x^2, y = \cos u \). Then \( (d/\!dx)(\cos x^2) = -2x \sin x^2 \).

(e) Let \( u = \tan x, y = u^2 \). Then \( d/\!dx)(\tan^2 x) = 2 \sec^2 x \tan x \).

(f) Let \( u = x^2, y = \tan u \). Then \( (d/\!dx)(\tan x^2) = 2x \sec^2 x \).

(g) Let \( u = 1/x, y = \cos u \). Then \( (d/\!dx)(\cos(1/x)) = 2 \sin(1/x)/x^2 \).

(h) Let \( u = -x, y = e^u \). Then \( (d/\!dx)(e^{-x}) = -e^{-x} \).

(i) Let \( u = 1/(x + 1), y = u^5 \). Then \( (d/\!dx)(1/(x + 1)^5) = -5/(x + 1)^6 \).

(j) Let \( u = x^3 + 1, y = u^4 \). Then \( (d/\!dx)(x^3 + 1)^4 = 12x^2(x^3 + 1)^3 \).

(k) Let \( u = 3x, y = \sin u \). Then \( (d/\!dx)(\sin 3x) = 3 \cos 3x \).

(l) Let \( u = \frac{1}{x}, y = \cos u \). Then \( (d/\!dx)(\cos \frac{1}{x}) = -\frac{1}{x^2} \sin \frac{1}{x} \).

(m) Let \( u = \frac{1}{x}, y = \tan u \). Then \( (d/\!dx)(\tan \frac{1}{x}) = \frac{1}{x^2} \sec^2 x \).

(n) Let \( u = -3x, y = e^u \). Then \( (d/\!dx)(e^{-3x}) = -3e^{-3x} \).

(o) Let \( u = 2x + 1, y = \sin u \). Then \( (d/\!dx)(\sin(2x + 1)) = 2 \cos(2x + 1) \).

(p) Let \( u = 3x - 2, y = \cos u \). Then \( (d/\!dx)(\cos(3x - 2)) = -3 \sin(3x - 2) \).

(q) Let \( u = 1 - 2x, y = \tan u \). Then \( (d/\!dx)(\tan(1 - 2x)) = -2 \sec^2(1 - 2x) \).

(r) Let \( u = 1/x, y = e^u \). Then \( (d/\!dx)(e^{1/x}) = -e^{1/x}/x^2 \).

3.5. All these problems use the result that \( (d/\!dx)x^\alpha = \alpha x^{\alpha - 1} \).

(a) \[ \frac{d}{dx}(x^{-2}) = -2x^{-3}. \]

(b) \[ \frac{d}{dx}(x^{-1}) = -\frac{1}{x^2}. \]
(c) \[ \frac{d}{dx} x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}}. \]

(d) \[ \frac{d}{dx} (x^{\frac{1}{3}}) = -\frac{1}{3} x^{-\frac{4}{3}}. \]

(e) \[ \frac{d}{dx} (x^{\frac{1}{3}}) = \frac{3}{2} x^{\frac{1}{2}}. \]

(f) \[ \frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}}. \]

(g) \[ \frac{d}{dx} x^3 = \frac{d}{dx} (x^{\frac{1}{3}}) = \frac{3}{2} x^{\frac{1}{2}}. \]

(h) \[ \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = -\frac{1}{x^2}. \]

(i) \[ \frac{d}{dx} \left[ \frac{1}{\sqrt{x}} \right] = \frac{d}{dx} (x^{-\frac{1}{2}}) = -\frac{1}{2x^{\frac{3}{2}}}. \]

3.6. (a) \[ \frac{d}{dx} (x^{\frac{1}{2}} \sin x) = x^{\frac{1}{2}} \cos x + \frac{\sin x}{2x^{\frac{1}{2}}}. \]

(b) \[ \frac{d}{dx} (\sin^4 x) = \frac{\cos x}{3 \sin^{\frac{3}{2}} x}. \]

(c) \[ \frac{d}{dx} [(x^2 + 1)^{-\frac{1}{2}}] = -\frac{x}{(x^2 + 1)^{\frac{3}{2}}}. \]

(d) \[ \frac{d}{dt} [\sin^2(3t + 1)] = 6 \cos(3t + 1) \sin(3t + 1). \]

(e) \[ \frac{d}{dt} (e^{-t} \cos t) = -e^{-t} (\cos t + \sin t). \]

(f) \[ \frac{d}{dt} (e^{-t} \sin t) = e^{-t} (\cos t - \sin t). \]

(g) \[ \frac{d}{dt} (e^{-2t} \cos 3t) = -e^{-2t} (2 \cos 3t + 3 \sin 3t). \]

(h) \[ \frac{d}{dt} (e^{-3t} \cos 2t) = -e^{-3t} (3 \cos 2t + 2 \sin 2t). \]

(i) \[ \frac{d}{dx} (\sin x \cos^2 x) = \cos^3 x - 2 \cos x \sin^2 x. \]
\( \frac{d}{dx} (\sin^2 x \cos x) = 2 \cos^2 x \sin x - \sin^3 x. \)

\( \frac{d}{dx} \left[ \left( \frac{\sin x}{x} \right)^2 \right] = \frac{2 \cos x \sin x}{x^2} - \frac{2 \sin^2 x}{x^3}. \)

\( \frac{d}{dx} [x(\sin^3 x)] = 3x \cos x \sin^2 x + \sin^3 x. \)

\( \frac{d}{dx} [x(\cos^3 x)] = -3x \cos^2 x \sin x + \cos^3 x. \)

3.7. (a) \( \frac{d}{dt} (\cos^2 x) = \frac{d}{dx} \frac{1}{2} (1 + \cos 2x) = -\sin 2x. \)

\( \frac{d}{dx} (\sin^2 x) = \frac{d}{dx} \frac{1}{2} (1 - \cos 2x) = \sin 2x. \)

(b) \( \frac{d}{dt} (\cos^2 x) = \frac{d}{dx} (\cos x \cos x) = -\cos x \sin x - \sin x \cos x = -\sin 2x. \)

\( \frac{d}{dx} (\sin^2 x) = \frac{d}{dx} (\sin x \sin x) = \sin x \cos x + \cos x \sin x = -\sin 2x. \)

(c) To apply the chain rule let \( u = \cos x, \ y = u^2. \) Then

\[ \frac{d}{dx} (\cos^2 x) = \frac{d}{dx} (\cos x) \frac{d}{dx} (\cos x) = -2 \cos x = -2 \cos x \sin x = -\sin 2x. \]

Let \( u = \sin x \) then

\[ \frac{d}{dx} (\sin^2 x) = \frac{d}{du} (u^2) \frac{d}{dx} (\sin x) = 2 \cos x = 2 \sin x \cos x = \sin 2x. \]

3.8.

(a) \( \frac{d^2 x}{dt^2} + 4x = (-4A \cos 2t - 4B \sin 2t) + 4(A \cos 2t + B \sin 2t) = 0 \)

(b) \( \frac{d^2 x}{dt^2} + n^2 x = (-n^2 A \cos nt - n^2 B \sin nt) + n^2 (A \cos nt + B \sin nt) = 0. \)

(c) \( \frac{d^2 x}{dt^2} - 9x = (9A e^{nt} + 9B e^{-nt}) - 9(A e^{nt} + B e^{-nt}) = 0. \)

(d) \( \frac{d^2 x}{dt^2} - n^2 x = (n^2 A e^{nt} + n^2 B e^{-nt}) - n^2 (A e^{nt} + B e^{-nt}) = 0. \)

(e) \( \frac{dx}{dt} = (-A + B)e^{-t} \cos t - (A + B)e^{-t} \sin t. \)

\[ \frac{d^2 x}{dt^2} = -2Be^{-t} \cos t + 2Ae^{-t} \sin t. \]
Hence
\[
\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = \\
[-2Be^{-t}\cos t + 2Ae^{-t}\sin t] + \\
2(-A + B)e^{-t}\cos t - (A + B)e^{-t}\sin t + 2[Ae^{-t}\cos t + Be^{-t}\sin t] = 0.
\]

(f) The fourth derivative of each term in \( y \) returns the same function in each case. Hence
\[
\frac{d^4y}{dx^4} - y = 0.
\]

3.9. Use the chain rule in the form
\[
\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{du} \frac{du}{dx}.
\]
The intermediate variables are defined in each problem.

(a) Let \( u = \cos x \), \( v = u^2 \), so that \( y = e^v \). Hence
\[
\frac{dy}{dx} = e^v \cdot 2u \cdot (-\sin x) = -2e^{\cos^2 x} \cos x \sin x.
\]

(b) Let \( u = x^2 \), \( v = \cos u \), so that \( y = e^{-v} \). Hence
\[
\frac{dy}{dx} = (-e^{-v}) \cdot (-\sin u) \cdot 2x = 2x \sin(x^2) e^{-\cos x^2}.
\]

(c) Let \( u = x^2 \), \( v = \cos u \), so that \( y = \ln v \). Hence
\[
\frac{dy}{dx} = \frac{1}{v} \cdot (-\sin u) \cdot 2x = -2x \tan(x^2).
\]

(d) Let \( u = x^2 \), \( v = e^u - 1 \), so that \( y = v^4 \). Hence
\[
\frac{dy}{dx} = 4v^3 \cdot e^u \cdot 2x = 8xe^{x^2}(e^{x^2} - 1)^3.
\]

3.10. Use the result (3.7) which states that if \( y = u(x)v(x)w(x) \) then \( \ln y = \ln u + \ln v + \ln w \) which when differentiated gives
\[
\frac{dy}{dx} = \frac{vw}{du} \frac{du}{dx} + \frac{wu}{dv} \frac{dv}{dx} + \frac{uw}{dw} \frac{dw}{dx}.
\]

(a) Let \( u = x \), \( v = e^x \), \( w = \sin x \). Then
\[
\frac{dy}{dx} = e^x \cdot \sin x \cdot 1 + \sin x \cdot e^x \cdot x + x \cdot e^x \cdot \cos x = e^x [\sin x + x \sin x + x \cos x].
\]

(b) Different variables are used. Let \( x = te^t \cos t \), and let \( u = t \), \( v = e^t \), \( w = \cos t \). Hence
\[
\frac{dx}{dt} = e^t \cdot \cos t + \cos t \cdot e^t + t \cdot e^t \cdot (-\sin t) = e^t [\cos t + t \cos t - t \sin t].
\]

(c) Let \( u = x^{\frac{1}{2}} \), \( v = e^{2x} \), \( w = \sin^{\frac{1}{2}} 3x \). Then
\[
\frac{dy}{dx} = e^{2x} \cdot \sin^{\frac{1}{2}} 3x \cdot \left[\frac{1}{2} e^{-\frac{x}{2}} + \sin^{\frac{1}{2}} 3x \cdot x^{\frac{1}{2}} \cdot 2e^{2x} + x^{\frac{1}{2}} \cdot e^{2x} \cdot \frac{3}{2} \cos 3x \cdot \sin^{\frac{1}{2}} 3x \right] = \frac{e^{2x}}{2x^{\frac{1}{2}} \sin^{\frac{1}{2}} x} [3x \cos 3x + (4x + 1) \sin 3x].
\]
This function and its derivative will only be real for restricted values of $x$—for example, for $0 \leq x \leq \frac{1}{2} \pi$.

3.11. (a) Treating $y$ as a function of $x$, and using the chain rule for $y(x)$,

$$\frac{dy}{dx}(x^2 + y^2) = 0,$$

or $2x + 2y \frac{dy}{dx} = 0.$

Hence $\frac{dy}{dx} = -x/y$ as required. Solving the equation $x^2 + y^2 = 4$ for $y$, it follows that $y = \pm(4 - x^2)^{\frac{1}{2}}$. Differentiating with respect to $x$, we have

$$\frac{dy}{dx} = \mp x(4 - x^2)^{-\frac{1}{2}} = \mp \frac{x}{y},$$

after substitution back in terms of $y$. This agrees with the answer obtained by the implicit method. Note that will always be two points on a circle which have the same slope. The tangent is always perpendicular to the radius to the point which has slope $m = y/x$.

(b) In this case for $x > 0$, $y \geq 0$,

$$\frac{dy}{dx}(x^2 + y^2) = 0,$$

or $\frac{1}{2}x^2 + \frac{1}{2}y^2 \frac{dy}{dx} = 0.$

Hence

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}.$$

(c) In this case

$$\frac{dy}{dx}(x^3 + xy - y^3) = 0$$

or $3x^2 + y + x \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0.$

Hence

$$\frac{dy}{dx} = -\frac{y + 3x^2}{x - 3y^2}.$$

(d) In this case

$$\frac{dy}{dx}(x \sin y - y \sin x) = 0$$

or $\sin y + x \cos y \frac{dy}{dx} - \frac{dy}{dx} \sin x - y \cos x = 0.$

Hence

$$\frac{dy}{dx} = \frac{y \cos x - \sin y}{x \cos y - \sin x}.$$

3.12. The expression for $\frac{dy}{dx}$ obtained from the implicit relation $f(x, y) = c$ does not depend on $c$. For example for $x^2 + y^2 = c$, we always have $\frac{dy}{dx} = -x/y$. However, the value of $\frac{dy}{dx}$ will depend indirectly on $c$ since $x$ and $y$ must always satisfy $x^2 + y^2 = c$.

3.13. If $xy^2 - x^2 y = 1$, then

$$y^2 + 2xy \frac{dy}{dx} - 2xy - x^2 \frac{dy}{dx} = 0.$$ (i)

Hence

$$\frac{dy}{dx} = \frac{2xy - y^2}{2xy - x^2}.$$ (ii)

Differentiate (i) again with respect to $x$:

$$2y \frac{dy}{dx} + 2y \frac{dy}{dx} + 2x \left( \frac{dy}{dx} \right)^2 + 2xy \frac{d^2y}{dx^2} - 2y - 2x \frac{dy}{dx} - 2x \frac{dy}{dx} - x^2 \frac{d^2y}{dx^2} = 0.$$ (iii)

Eliminate $\frac{dy}{dx}$ between (ii) and (iii): the answer is

$$\frac{d^2y}{dx^2} = \frac{6xy(-x^3 + 2x^2 y - 2xy^2 + y^3)}{(2xy - x^2)^3}.$$
3.14. (a) Let $y = \arcsin x$. Then $x = \sin y$. Differentiating with respect to $y$,

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Hence

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sqrt{1 - x^2}}.$$

(b) Let $y = \arccos x$ so that $x = \cos y$. Then

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - x^2}}.$$

(c) Let $y = \arctan x$ so that $x = \tan y$. Then

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}.$$

(d) Let $y = \sinh^{-1} x$ so that $x = \sinh y$. Then

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + x^2}}.$$

(e) Let $y = \cosh^{-1} x$ so that $x = \cosh y$. Then

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sinh y} = \frac{1}{\sqrt{(x^2 - 1)}}.$$

(f) Let $y = \tanh^{-1} x$ so that $x = \tanh y$. Then

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sech^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}.$$

3.15. (a) If $r = \sin \frac{1}{2} \theta$, then the $(x, y)$ coordinates are

$$x = r \cos \theta = \sin \frac{1}{2} \theta \cos \theta, \quad y = r \sin \theta = \sin \frac{1}{2} \theta \sin \theta.$$

Using parametric differentiation,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{1}{2} \cos \frac{1}{2} \theta \sin \theta + \sin \frac{1}{2} \theta \cos \theta}{\frac{1}{2} \cos \frac{1}{2} \theta \cos \theta - \sin \frac{1}{2} \theta \sin \theta}.$$

At $\theta = \frac{1}{4} \pi$,

$$\frac{dy}{dx} = \frac{\frac{1}{2} \sqrt{2} + 0}{0 - \frac{1}{2}} = -\frac{1}{2}.$$

(b) If $r = 1 + \sin^2 \theta$, then

$$x = r \cos \theta = (1 + \sin^2 \theta) \cos \theta, \quad y = r \sin \theta = (1 + \sin^2 \theta) \sin \theta.$$

Hence

$$\frac{dx}{d\theta} = -\sin \theta - \sin^3 \theta + 2 \cos^2 \theta \sin \theta = -\frac{1}{2 \sqrt{2}} \quad \text{at} \quad \theta = \frac{1}{4} \pi,$$

and

$$\frac{dy}{d\theta} = \cos \theta + 3 \sin^2 \theta \cos \theta = \frac{1}{\sqrt{2}} + \frac{3}{2 \sqrt{2}} = \frac{5}{2 \sqrt{2}} \quad \text{at} \quad \theta = \frac{1}{4} \pi.$$
Hence

\[ \frac{dy}{dx} = \frac{dy}{d\theta} \left/ \frac{dx}{d\theta} \right. = -5. \]

3.16. (a) For \( x = t^3 \) and \( y = t^2 \),

\[ \frac{dy}{dx} = \frac{dy}{dt} \left/ \frac{dx}{dt} \right. = \frac{2t}{3t^2} = \frac{2}{3t^{\frac{2}{3}}}. \]

(b) For \( x = 2 \cos t \) and \( y = 2 \sin t \),

\[ \frac{dy}{dx} = \frac{dy}{dt} \left/ \frac{dx}{dt} \right. = \frac{2 \cos t}{-2 \sin t} = \pm \frac{x}{2\sqrt{4-x^2}}, \]

assuming \( 0 \leq t \leq \frac{1}{2} \pi \).

3.17. Elimination of \( t \) between \( x \) and \( y \), using the identity \( \cos^2 t + \sin^2 t = 1 \), gives the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]

The derivative is given by

\[ \frac{dy}{dx} = \frac{dy}{dt} \left/ \frac{dx}{dt} \right. = \frac{b \cos t}{-a \sin t} = -\frac{b}{a} \cot t. \]

The speed of the point is

\[ \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]^\frac{1}{2} = \sqrt{[a^2 \sin^2 t + b^2 \cos^2 t]}. \]

3.18. In exponential form \( a^x = e^{x \ln a} \). Hence

\[ \frac{d}{dx} (a^x) = (\ln a) e^{x \ln a} = x^a \ln a. \]

Chapter 4: Applications of differentiation

4.1. \( f(u) = u^2 \), so \( f'(u) = 2u \) and \( f''(u) = 2 \) for all arguments \( u \).

(a) Let \( u = t \). Then \( f'(t) = 2t \).

(b) Let \( u = t^2 \). Then \( f'(t^2) = 2u = 2t^2 \).

(c) \( (d/dt)f(t^2) = (d/dt)t^4 = 4t^3 \).

(d) Let \( u = t^{\frac{1}{2}} \). Then \( f'(t^{\frac{1}{2}}) = 2u = 2t^{\frac{1}{2}} \).

(e) \( (d/dt)f(t^{\frac{1}{2}}) = (d/dt)t = 1 \).

(f) Let \( u = t^{\frac{1}{3}} \). Then \( f''(t^{\frac{1}{3}}) = 2 \).

4.2. Denote the function in each case by \( f(x) \). The stationary points are given by \( f'(x) = 0 \). If \( A: x = a \) is a stationary point, then \( A \) is a minimum if \( f''(a) > 0 \), or a maximum if \( f''(a) < 0 \). If \( f''(a) = 0 \), then the stationary point can be a minimum, maximum or point of inflection depending on the sign of \( f'(x) \) on either side of \( x = a \).

(a) Since \( f(x) = x^2 - x \), then

\[ f'(x) = 2x - 1, \quad f''(x) = 2. \]

Stationary point: \( x = \frac{1}{2} \).
Test: $f''(\frac{1}{2}) = 2 > 0$ so $x = \frac{1}{2}$ is a minimum.
(b) Since $f(x) = x^2 - 2x - 3$, then
\[ f'(x) = 2x - 2, \quad f''(x) = 2. \]
Stationary point: $x = 1$.
Test: $f''(1) = 2 > 0$ so $x = 1$ is a minimum.
(c) Since $f(x) = x\ln x$, then
\[ f'(x) = 1 + \ln x, \quad f''(x) = \frac{1}{x}. \]
Stationary point: $x = e^{-1}$.
Test: $f''(e^{-1}) = e > 0$ so $x = e^{-1}$ is a minimum.
(d) Since $f(x) = xe^{-x}$, then
\[ f'(x) = (1 - x)e^{-x}, \quad f''(x) = (-2 + x)e^{-x}. \]
Stationary point: $x = 1$.
Test: $f''(1) = -e^{-1} < 0$ so $x = 0$ is a maximum.
(e) Since $f(x) = 1/(x^2 + 1)$, then
\[ f'(x) = \frac{-2x}{(x^2 + 1)^2}, \quad f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}. \]
Stationary point: $x = 0$.
Test: $f''(0) = -2 < 0$ so $x = 0$ is a maximum.
(f) Since $f(x) = x^2 - 3x + 2$, then
\[ f'(x) = 2x - 3, \quad f''(x) = 2. \]
Stationary point: $x = \frac{3}{2}$.
Test: $f''(\frac{3}{2}) = 2 > 0$ so $x = \frac{3}{2}$ is a minimum.
(g) Since $f(x) = e^x + e^{-x}$, then
\[ f'(x) = e^x - e^{-x}, \quad f''(x) = e^x + e^{-x}. \]
Stationary point: $x = 0$.
Test: $f''(0) = 2 > 0$ so $x = 0$ is a minimum.
(h) Since $f(x) = x^2 + 4x + 2$, then
\[ f'(x) = 2x + 4, \quad f''(x) = 2. \]
Stationary point: $x = -2$.
Test: $f''(-2) = 2 > 0$ so $x = -2$ is a minimum.
(i) Since $f(x) = x - x^3$, then
\[ f'(x) = 1 - 3x^2, \quad f''(x) = -6x. \]
Stationary points: $x = \pm 1/\sqrt{3}$.
Tests: $f''(1/\sqrt{3}) = -6/\sqrt{3} < 0$, so $x = 1/\sqrt{3}$ is a maximum.
$f''(-1/\sqrt{3}) = 6/\sqrt{3} > 0$ so $x = -1/\sqrt{3}$ is a minimum.
(j) Since $f(x) = x^2(x - 1)$, then
\[ f'(x) = 3x^2 - 2x, \quad f''(x) = 6x - 2. \]
Stationary points: $x = 0$ and $x = \frac{2}{3}$.
Tests: $f''(0) = -2 < 0$ so $x = 0$ is a minimum.
$f''(1) = 2 > 0$ so $x = \frac{1}{2}$ is a minimum.

(k) Since $f(x) = \sin x - \cos x$, then
$$f'(x) = \cos x + \sin x, \quad f''(x) = -\sin x + \cos x.$$

Stationary points: $x = \frac{3}{4} \pi$ and $\frac{7}{4} \pi$ for $0 < x < 2\pi$.
Tests: $f''(\frac{3}{4} \pi) = -\sqrt{2} < 0$ so $x = \frac{3}{4} \pi$ is a maximum;
$f''(\frac{7}{4} \pi) = \sqrt{2} > 0$ so $x = \frac{7}{4} \pi$ is a minimum.

(l) Since $f(x) = \sin x \cos x - \frac{1}{2} \sin 2x$, then
$$f'(x) = \cos 2x, \quad f''(x) = -2\sin 2x.$$

Stationary points occur where $\sin 2x = 0$ or $\cos 2x = 0$ for $-\pi < x < \pi$.
Tests: $f''(-\frac{1}{4} \pi) = -2 < 0$ so $x = -\frac{1}{4} \pi$ is a maximum;
$f''(\frac{1}{4} \pi) = 2 > 0$ so $x = \frac{1}{4} \pi$ is a minimum;
$f''(\frac{1}{2} \pi) = 2 > 0$ so $x = \frac{1}{2} \pi$ is a minimum.

(m) Since $f(x) = e^{-x} \sin x$, then
$$f'(x) = e^{-x} (-\sin x + \cos x), \quad f''(x) = -2e^{-x} \cos x.$$

Stationary points occur where $\tan x = 1$, at $x = (n + \frac{1}{2}) \pi$, $(n = 0, \pm 1 \pm 2, \ldots)$. Tests: $f''[(n + \frac{1}{2}) \pi] = -\sqrt{2} (-1)^n e^{-(n + \frac{1}{2}) \pi} < 0$ or $> 0$ according as $n$ is even or odd. Hence the stationary point is a maximum $x = (n + \frac{1}{2}) \pi$ is a maximum if $n$ is even, and a minimum if $n$ is odd.

(n) Since $f(x) = e^{-\frac{1}{2} x} \sin 2x$, then
$$f'(x) = \frac{1}{2} e^{-\frac{1}{2} x} (-\sin 2x + 6 \cos 2x), \quad f''(x) = \frac{1}{4} e^{-\frac{1}{2} x} (-12 \cos 2x - 35 \sin 2x).$$

Stationary points occur where $\tan 2x = 6$ at $x = \alpha + \frac{1}{2} n \pi$, $(n = 0, \pm 1 \pm 2, \ldots)$, where $\alpha = \frac{1}{4} \arctan 6$.
Tests: $f''(\alpha + \frac{1}{2} n \pi) = -222e^{-\frac{1}{2} (\alpha + \frac{1}{2} n \pi)} (-1)^n / [9 \sqrt{37}] < 0$ or $> 0$ according as $n$ is even or odd. Hence the stationary point $x = \alpha + \frac{1}{2} n \pi$ is a maximum if $n$ is even, and a minimum if $n$ is odd.

(o) Since $f(x) = x - \cos x$, then
$$f'(x) = 1 + \sin x, \quad f''(x) = \cos x.$$

Stationary points occur where $\sin x = -1$ at $x = (2n - \frac{1}{2}) \pi$, $(n = 0, \pm 1 \pm 2, \ldots$).
Tests: $f''[(2n - \frac{1}{2}) \pi] = 0$ for all $n$. Hence the test fails. But $f'(x) = 1 + \sin x \geq 0$ for all $x$. Hence all the stationary points must be points of inflection.

(p) Since $f(x) = 2e^x - \frac{1}{2} e^{2x}$, then
$$f'(x) = 2e^x - e^{2x}, \quad f''(x) = 2e^x - 2e^{2x}.$$

Stationary point occurs where $e^x = 2$ at $x = \ln 2$.
Test: $f''(\ln 2) = -4 < 0$ so $x = \ln 2$ is a maximum.

(q) Since $f(x) = x^2 e^{-x}$, then
$$f'(x) = xe^{-x}(2 - x), \quad f''(x) = e^{-x}(2 - 4x + x^2).$$

Stationary points at $x = 0$ and $x = 2$.
Tests: $f''(0) = 2 > 0$ so that $x = 0$ is a minimum; $f''(2) = -2e^{-2} < 0$ so that $x = 2$ is a maximum.

(r) Since $f(x) = (\ln x)/x$, then
$$f'(x) = \frac{1}{x^2} (1 - \ln x), \quad f''(x) = \frac{1}{x^3} (-3 + 2 \ln x).$$
Stationary point where \( \ln x = 1 \) at \( x = e \).
Test: \( f''(e) = -e^{-3} < 0 \) so \( x = e \) is a maximum.
(s) Since \( f(x) = (1 - x)^3 \), then
\[
f'(x) = -3(1 - x)^2, \quad f''(x) = 6(1 - x)^2.
\]
Stationary point: \( x = 1 \).
Test: \( f''(1) = 0 \) and the test fails. However, \( f'(x) \leq 0 \) for all \( x \). Hence \( x = 1 \) is a point of inflection.
(t) Since \( f(x) = \sin x \), then
\[
f'(x) = 3 \sin^2 x \cos x, \quad f''(x) = 3 \sin x(2 - 3 \sin^2 x).
\]
Stationary points occur where \( \sin x = 0 \), at \( x = n\pi \), and where \( \cos x = 0 \), at \( x = (n + \frac{1}{2})\pi \) for \( n = 0, \pm 1, \pm 2, \ldots \).
Tests: \( f''(n\pi) = 0 \) so that the test fails. However, if \( n \) is even, then \( f'(x) \) is positive in a small interval including \( x = n\pi \), and if \( n \) is odd, then \( f'(x) \) is negative in a small interval including \( x = n\pi \). In both cases, therefore, the stationary point is a point of inflection.
\[
f''[n + \frac{1}{2} \pi] = -3(-1)^n \] so that \( x = (n + \frac{1}{2})\pi \) is a maximum if \( n \) is even, and a minimum if \( n \) is odd.
(u) Since \( f(x) = e^{-x^2} \), then
\[
f'(x) = -2xe^{-x^2}, \quad f''(x) = 2e^{-x^2}(2x^2 - 1).
\]
Stationary point: \( x = 0 \).
Test: \( f''(0) = -2 < 0 \) so \( x = 0 \) is a maximum.
(v) Since \( f(x) = e^{x^2 - x} \), then
\[
f'(x) = (2x - 1)e^{x^2 - x}, \quad f''(x) = (4x^2 - 4x + 3)e^{x^2 - x}.
\]
Stationary point: \( x = \frac{1}{2} \).
Test: \( f''(\frac{1}{2}) = 2e^{-\frac{3}{4}} > 0 \) so \( x = \frac{1}{2} \) is a minimum.
(w) Since \( f(x) = x + x^{-1} \), then
\[
f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.
\]
Stationary points: \( x = \pm 1 \).
Tests: \( f''(1) = 2 > 0 \) so \( x = 1 \) is a minimum; and \( f''(-1) = -2 < 0 \) so \( x = -1 \) is a maximum.
(x) Since \( f(x) = x^3e^{-x} \), then
\[
f'(x) = x^2e^{-x}(3 - x), \quad f''(x) = xe^{-x}(6 - 6x + x^2).
\]
Stationary points: \( x = 0 \) and \( x = 3 \).
Tests: \( f''(0) = 0 \) and the test fails. In a small interval which includes the origin \( f'(x) > 0 \) which means that \( x = 0 \) is a point of inflection.
\( f''(3) = -9e^{-3} < 0 \) so \( x = 3 \) is a maximum.

4.3. If \( y = f[u(x)] \), then by the chain rule
\[
\frac{dy}{dx} = \frac{df}{du} \frac{du}{dx} = f'(u)u'(x),
\]
and
\[
\frac{d^2y}{dx^2} = \frac{d}{dx}[f'(u)u'(x)] = \frac{df}{du} \frac{du}{dx}[f'(u)]u'(x) + f'(u)u''(x) \quad \text{(product rule)}
\]
\[
= f''(u)[u'(x)]^2 + f'(u)u''(x)
\]

35
Since $f'(u) > 0$ for all $u$, then $dy/dx$ can only be zero if $u'(x) = 0$ (by the chain rule (3.3)). Hence $f[u(x)]$ and $u(x)$ have stationary points only at the same values of $x$.

In 4.2(v), $f(u) = e^u$ and $u = x^2 - x$.

4.4. If the sides have lengths $x > 0$ and $y > 0$, then the given area $A = xy$. The length of the perimeter is $P = 2x + 2y$. Eliminate $y$ so that

$$P = 2x + \frac{2A}{x}.$$ 

The first and second derivatives of $P$ are

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2}, \quad \frac{d^2P}{dx^2} = \frac{4A}{x^3}.$$ 

Hence the perimeter length is stationary where $dP/dx = 0$: at $x = \pm\sqrt{A}$. Since $x > 0$, choose the stationary value $x = \sqrt{A}$. For this value

$$\frac{d^2P}{dx^2} = \frac{4A}{A^2} > 0,$$

so that the perimeter is a minimum when $x = y = \sqrt{A}$. The piece of ground must be a square.

4.5. Let the the base of the cross-section be $x > 0$ which will also be the diameter of the semicircle, and let the height of the rectangle be $y > 0$. The given area $A$ of the tunnel cross-section is

$$A = xy + \frac{1}{8}\pi x^2.$$ 

The length of the perimeter is $P = x + 2y + \frac{1}{2}\pi x$. Eliminate $y$, so that

$$P = \frac{2}{x} \left( A - \frac{1}{8}\pi x^2 \right) + x(1 + \frac{1}{2}\pi) = \frac{2A}{x} + (1 + \frac{1}{4}\pi)x.$$ 

This is stationary where

$$\frac{dP}{dx} = -\frac{2A}{x^2} + 1 + \frac{1}{4}\pi = 0,$$

which occurs at $x = \sqrt{\frac{4A}{\pi}}$ (choosing the positive root). The perimeter is a minimum since

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0$$

at the stationary point.

4.6. Let $r$ be the radius of the base and $h$ the height of the drum. The volume $V$ of the drum is given by $V = \pi r^2 h$ and its prescribed surface area by $A = 2\pi r^2 + 2\pi rh$. We are given that $A$ is a constant, so eliminate $h$ in the expression for $V$:

$$V = \frac{1}{2}[Ar - 2\pi r^3].$$ 

Differentiating

$$\frac{dV}{dr} = \frac{1}{2}[A - 6\pi r^2], \quad \frac{d^2V}{dr^2} = -6\pi r.$$ 

The volume is stationary where

$$\frac{dV}{dr} = 0, \text{ at } r = \sqrt{\frac{A}{6\pi}}$$

choosing the positive root. Obviously $d^2V/dr^2 < 0$ which proves that this radius gives a minimum volume. The height of this drum is $h = \sqrt{\frac{2A}{(3\pi)}}$ which is equal to its diameter.
4.7. Similar to 4.6: the volume is given by the same formula but the prescribed $A$ is different:

$$V = \pi r^2 h, \quad A = \pi r^2 + 2\pi rh.$$  

Elimination of $h$ leaves

$$V = \frac{1}{2} r(A - \pi r^2).$$

Differentiating

$$\frac{dV}{dr} = \frac{1}{2} [A - 3\pi r^2], \quad \frac{d^2V}{dr^2} = 3\pi r.$$  

Hence the radius and height of the drum of minimum volume are

$$r = h = \sqrt[3]{\frac{A}{3\pi}}.$$  

4.8. (a) $y = 1/(x^2 + 1)$:

![Figure 36: Problem 4.8a](image)

(b) $y = e^{x^2}$:

![Figure 37: Problem 4.8b](image)

(c) $y = x/(x - 1)$:

![Figure 38: Problem 4.8c](image)

(d) $y = xe^{-x}$;
(e) $y = x^2e^{-x}$;
(f) $y = xe^{-x}$;
(g) $y = e^{2x} - 4e^x$;
(h) $y = (\ln x)/x$ for $x > 0$:
(i) $[\ln(-x)]/x$ for $x < 0$:
(j) $y = x \ln x - x$ for $x > 0$. 

37
(k) $y = \sin(1/x)$;  
(l) $y = (x^2 - 1)^2$;  
(m) $y = x(x^2 - 1)^2$;  
(n) $y = (\sin x)/x$.

4.9. (a) $y = 1/(x^2 - 1)$;  
(b) $y = x/(x^2 - 1)$;  
(c) $1/[x(x - 2)]$;  
(d) $y = x^3/(1 - x)$;  
(e) $y = (x + 2)/(x - 1)$;  
(f) $y = 1/(x + 1) + 1/(x + 2)$:

4.10. The incremental formula given by (4.4) is

$$\delta y \approx f'(a)\delta x \text{ at } x = a.$$  

The exact value is given by

$$\delta y = f(a + \delta x) - f(a).$$

(a) $f(x) = x^3$: $\delta y \approx 3x^2\delta x$. With $x = 2$ and $\delta x = 0.1$, the approximate and exact values are given by

$$\delta y \approx 1.200, \quad \delta y = (2.1)^3 - 2^3 = 1.157\ldots.$$  

(b) $f(x) = x\sin x$: $\delta y \approx (\sin x + x \cos x)\delta x$. With $x = \frac{1}{2}\pi$ and $\delta x = -0.2$ the approximate and exact values are given by

$$\delta y \approx (\sin \frac{1}{2}\pi + \frac{1}{2}\pi \cos \frac{1}{2}\pi)(-0.2) = -0.2,$$

$$\delta y = (\frac{1}{2}\pi - 0.2)\sin(\frac{1}{2}\pi - 0.2) - \frac{1}{2}\pi\sin \frac{1}{2}\pi = -0.227\ldots.$$
(c) $f(x) = \cos x$: $\delta y \approx -\sin x \delta x$. With $x = \frac{1}{4} \pi$ and $\delta x = 0.1$ the approximate and exact values are given by

$$\delta y \approx (-\sin \frac{1}{4} \pi)(0.1) = -0.0707\ldots,$$

$$\delta y = \cos(\frac{1}{4} \pi + 0.1) - \cos(\frac{1}{4} \pi) = -0.0741\ldots.$$

(d) $f(x) = (1 + x)/(1 - x)$: $\delta y = 2/(1 - x)^2 \delta x$. With $x = 2$ and $\delta x = -0.2$ the approximate and exact values are given by

$$\delta y \approx \frac{2}{(1 - 2)^2}(-0.2) = -0.4, \quad \delta y = -0.5.$$

(e) $y = \tan x$: $\delta y \approx \sec^2 x \delta x$. With $x = \frac{1}{4} \pi$ and $\delta x = 0.1$ the approximate and exact values are given by

$$\delta y \approx (\sec^2 \frac{1}{4} \pi)(0.1) = 0.2, \quad \delta y = \tan(\frac{1}{4} \pi + 0.1) - \tan \frac{1}{4} \pi = 0.223\ldots.$$

(f) $f(x) = 1/(1 - x^2)$: $f'(x) = 2x/(1 - x^2)^2$. With $x = 0.5$ and $\delta x = \pm 0.1$ the approximate and exact values are given by

$$\delta y \approx \frac{1}{(1 - 0.5^2)^2}(\pm 0.1) = \pm 0.177\ldots,$$

$$\delta y = \frac{1}{1 - (0.05 \pm 0.1)^2} - \frac{1}{1 - (0.05)^2} = 0.229\ldots \text{ or } -0.142\ldots.$$

4.11. (a) With $f$ fixed,

$$\frac{dv}{du} = -\frac{f^2}{(u - f)^2}.$$
Hence, with $f = 0.75$, $u = 1.25$ and $\delta u = 0.05$,

$$
\delta v \approx -\frac{f^2 \delta u}{(u - f)^2} = \frac{-(0.75)^2(0.05)}{(1.25 - 0.75)^2} = -0.112 \ldots.
$$

(b) The voltage is given by

$$
v = \frac{E(R_1 R_4 - R_2 R_3)}{(R_1 + R_2)(R_3 + R_4)}.\]

Its derivative with respect to $R_1$ is

$$
\frac{dv}{dR_1} = \frac{E R_2}{(R_1 + R_2)^2}.
$$

Hence

$$
\delta v \approx \frac{E R_2}{(R_1 + R_2)^2} = \frac{5}{18} \delta R_1.
$$

(c) With $b$ and $A$ constant in $a = b \sin A / (\sin B)$,

$$
\frac{da}{dB} = \frac{-b \sin A \cos B}{\sin^2 B} = -a \cot B.
$$

Hence

$$
\delta a \approx -a \cot B \delta B.
$$

(d) In terms of $a, b, c$,

$$
A = \frac{1}{4} \sqrt{[(a + b + c)(-a + b + c)(a - b + c)(a + b - c)]}.
$$
Logarithmic differentiation (see equation (3.7)) gives

\[
\frac{1}{A} \frac{dA}{dc} = \frac{1}{2} \left[ \frac{1}{a+b+c} + \frac{1}{-a+b+c} + \frac{1}{a-b+c} + \frac{1}{a+b-c} \right].
\]

The incremental formula for \( \delta A \) becomes, at \( a = 2, b = 4, c = 5 \)

\[
\delta A \approx \frac{dA}{dc} \delta c = -\frac{25}{2\sqrt{231}} (0.1) = 0.0822\ldots.
\]

4.12. Given \( C = P(1 + r)^n \).
(a) With \( n \) and \( P \) fixed,

\[
\frac{dC}{dr} = Pn(1 + r)^{n-1},
\]
so that \( \delta C \approx Pn(1 + r)^{n-1} \delta r \).

(b) With \( r \) and \( P \) fixed,

\[
\frac{dC}{dn} = P \frac{d}{dn} e^{n\ln(1+r)} = P(1 + r)^n \ln(1 + r) \quad \text{see Problem 3.18.}
\]

Hence

\[
\delta C \approx P(1 + r)^n \ln(1 + r) \delta n.
\]

(c) Suppose that \( P = £100, r = 0.05 \) (5%) and \( n = 10 \) years. The tables below show comparisons between the approximate increments \( \delta C \) for decreasing values of \( \delta r \) (\( n \) fixed) and \( \delta n \) (\( r \) fixed).
\[ \delta r \approx \frac{C_n(1 + r)^n - C}{C(1 + r + \delta r)^n - C(1 + r)^n} \]

<table>
<thead>
<tr>
<th>( \delta r )</th>
<th>approximate increment in ( C )</th>
<th>exact increment in ( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>15.513...</td>
<td>16.195...</td>
</tr>
<tr>
<td>0.005</td>
<td>7.756...</td>
<td>7.925...</td>
</tr>
<tr>
<td>0.001</td>
<td>1.551...</td>
<td>1.557...</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.155...</td>
<td>0.155...</td>
</tr>
</tbody>
</table>

\[ \delta n \approx \frac{P_n(1 + r)^n \ln(1 + r) \delta n}{P(1 + r)^n - P(1 + r)^n} \]

<table>
<thead>
<tr>
<th>( \delta n )</th>
<th>approximate increment in ( C )</th>
<th>exact increment in ( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.947...</td>
<td>8.144...</td>
</tr>
<tr>
<td>0.1</td>
<td>0.795...</td>
<td>0.797...</td>
</tr>
<tr>
<td>0.01</td>
<td>0.080...</td>
<td>0.080...</td>
</tr>
</tbody>
</table>

4.13. The iterations in Newton’s method are

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \]

for a given initial value \( x_0 \). (a) Let \( f(x) = x^4 + 2x^2 - x - 1 \). Then

\[ f'(x) = 4x^3 + 4 - 1. \]

For example, if we start with \( x_0 = 0.75 \), we obtain

\[ x_1 = 0.75 - \frac{f(0.75)}{f'(0.75)} = 0.833 \ldots. \]
Repeat the process starting with $x_1$ to obtain $x_2$, and so on. The solution is $x = 0.825\ldots$

(b) Let $f(x) = x^4 + x^{1\over 7} - 1$. Then

$$f'(x) = 4x^3 + \frac{1}{3}x^{-{5\over 7}}.$$

The solution is $x = 0.619\ldots$

(c) Let $f(x) = x \ln x + 0.3$. Then

$$f'(x) = 1 + \ln x.$$

The solution is $x = 0.168\ldots$

(d) Let $f(x) = e^x - 4x^3$. Then

$$f'(x) = e^x - 12x^2.$$

The solution is $x = 0.831\ldots$

(e) Let $f(x) = \tan x - 2x$. Then

$$f'(x) = \sec^2 x - 2.$$

By Newton’s method the solution is $x = 1.165\ldots$ 4

(f) Let $f(x) = e^x \sin x/(1 + x)$. Then

$$x - \frac{f(x)}{f'(x)} = x + \frac{(1 + x)[2(1 + x)e^{-x} - \sin x]}{(1 + x)\cos x + x \sin x}.$$

Let $x_0 = 1.85$. Then

$$x_1 = 1.663, \quad x_2 = 1.689, \quad x_3 = 1.690$$

to three decimal places.

4.14. Since $f(x) = xe^{-x} + 1$ then $f'(x) = (1 - x)e^{-x}$. The function has its only stationary point at $x = 1$ which is a maximum. The slope of $y = f(x)$ in the neighbourhood of the solution of $f(x) = 0$ is therefore positive whilst that for any value of $x$ greater than 1 will be negative. By the geometrical construction of Newton’s method illustrated in Figure 4.15, any tangent which starts for $x > 1$ will produce iterations which diverge from the required solution. The graph of $y = xe^{-x} - 1$ is shown in the figure.

4.15. (a) The graph shows a continuous function in which $f(a)$ and $f(b)$ have opposite signs.

(b) Let $g(x) = e^x - 3x$. The table gives a sequence of values for $g(x)$ at intervals 0.25.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
<th>$0.25$</th>
<th>$0.5$</th>
<th>$0.75$</th>
<th>$1.0$</th>
<th>$1.25$</th>
<th>$1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>1</td>
<td>0.534</td>
<td>0.149</td>
<td>-0.133</td>
<td>-0.282</td>
<td>-0.260</td>
<td>-0.018</td>
</tr>
</tbody>
</table>
Evidently the solutions of the equation lie between \( x = 0.5 \) and \( x = 0.75 \), and between \( x = 1.5 \) and \( x = 1.75 \). Note also that the function has a minimum value at \( x = \ln 3 = 1.098 \ldots \), which means, for example, that any initial value for the smaller solution must start at a value of \( x < \ln 3 \) for the reasons outlined in Problem 4.14. Similar conditions apply to the other solution.

4.16. (a) Calculate \( f(a + nE) \) for \( n = 1, 2, \ldots \). Stop the program at \( n = N \), when \( f(a + NE) \) and \( f(a + (N - 1)E) \) have different signs.
(b) In the following table the interval is bisected four times with \( E = 0.125 \) and \( N = 8 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(a + NE) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.75</td>
<td>-1</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.983</td>
</tr>
<tr>
<td>2.25</td>
<td>-0.929</td>
</tr>
<tr>
<td>2.5</td>
<td>-0.829</td>
</tr>
<tr>
<td></td>
<td>-0.500</td>
</tr>
<tr>
<td></td>
<td>-0.676</td>
</tr>
<tr>
<td></td>
<td>-0.457</td>
</tr>
</tbody>
</table>

The solution of the equation lies between \( x = 0.75 \) and \( x = 0.875 \). The computed solution is \( x = 0.806 \ldots \).
(c) Four decimal accuracy is obtained after 10 iterations using the bisection method, whilst Newton’s method achieve the same accuracy after just 4 iterations.

4.17. The slope of the normal at \( x = x_0 \) is \(-1/f'(x_0)\) and at \( x = x_0 + \delta x_0 \) is \(-1/f'(x_0 + \delta x_0)\). Hence their equations are

\[
y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0),
\]

\[
y - f(x_0 + \delta x_0) = -\frac{1}{f'(x_0 + \delta x_0)}(x - x_0 - \delta x_0).
\]

Solving these equations for \( x \) and \( y \):

\[
x = x_0 - \frac{f'(x_0)[f'(x_0 + \delta x_0)[f(x_0 + \delta x_0) - f(x_0)] + \delta x_0]}{f'(x_0 + \delta x_0) - f'(x_0)},
\]

\[
y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0) + \frac{1}{f'(x_0 + \delta x_0)}(x - x_0 - \delta x_0).
\]
\[ y = f(x_0) + \frac{[f'(x_0 + \delta x_0)[f(x_0 + \delta x_0) - f(x_0)] + \delta x_0]}{f'(x_0 + \delta x_0) - f'(x_0)} \, . \]

Divide the numerators and denominators by \( \delta x_0 \) and let the increment tend to zero, so that the centre of curvature \((x_c, y_c)\) is located at

\[ (x_0 - \frac{f'(x_0)[1 + f'(x_0)^2]}{f''(x_0)}, f(x_0) + \frac{[1 + f'(x_0)^2]}{f''(x_0)}) \, . \]

The radius of curvature

\[ R = \sqrt{[(x_c - x_0)^2 + (y_c - y_0)^2]} = \frac{[1 + f'(x_0)^2]^2}{f''(x_0)}. \]

For the parabola \( y = x^2 \),

\[ f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2. \]

Hence the centre of curvature of the point \((x_0, x_0^2)\) is located at

\[ [x_0 - x_0(1 + 4x_0^2), x_0^2 + \frac{1}{2}(1 + 4x_0^2)], \]

and its radius of curvature is \( R = \frac{1}{2}(1 + 4x_0^2)^{\frac{3}{2}} \).

**4.18.** We shall prove Leibniz’s formula by induction. For \( n = 1 \), the formula is true since

\[ (fg)^{(1)} = f^{(1)}g + fg^{(1)} \]

by the product rule; note that \( 1C_1 = 1 \). Assume that the given formula is true for \( n = k \) and all \( x \). Then

\[ (fg)^{(k)} = f^{(k)}g + kC_1f^{(k-1)}g^{(1)} + kC_2f^{(k-2)}g^{(2)} + \cdots + kC_kfg^{(k)}. \]

Differentiate both sides with respect to \( x \):

\[ (fg)^{(k+1)} = (f^{(k+1)}g + f^{(k)}g^{(1)}) + (kC_1f^{(k)}g^{(1)} + kC_1f^{(k-1)}g^{(2)}) + (kC_2f^{(k-1)}g^{(2)} + kC_2f^{(k-2)}g^{(2)}) + \cdots + (kC_kfg^{(k)} + kC_kfg^{(k+1)}) \]

\[ = f^{(k+1)}g + (1 + kC_1)f^{(k)}g^{(1)} + (kC_1 + kC_2)f^{(k-1)}g^{(2)} + \cdots + kC_kfg^{(k+1)}. \]

The coefficients can be written as

\[ 1 + kC_1 = 1 + \frac{k!}{1!(k-1)!} = k + 1 = k+1C_1, \]

\[ kC_1 + kC_2 = \frac{k!}{1!(k-1)!} + \frac{k!}{2!(k+2)!} = k + \frac{k(k-1)}{2!} = \frac{k(k+1)}{2!} = k+1C_2, \]

and, in general,

\[ kC_r + kC_{r+1} = \frac{k!}{r!(k-r)!} + \frac{k!}{(r+1)!(k-r-1)!} = \frac{k!}{r!(k-r-1)!} \left[ \frac{1}{k-r+1} + \frac{1}{r+1} \right] = \frac{k!(k+1)}{r!(k-r-1)!(r+1)} = \frac{(k+1)!}{(r+1)!}(k-r)! = k+1C_{r+1}. \]

Hence

\[ (fg)^{(k+1)} = f^{(k+1)}g + k+1C_1f^{(k)}g^{(1)} + k+1C_2f^{(k-1)}g^{(2)} + \cdots + k+1C_{k+1}fg^{(k+1)}. \]
Hence if the result is true for \(n = k\), then it is true for \(n = k + 1\). We have shown that it is true for \(n = 1\) (the product rule); therefore it is true for \(n = 2, 3, \ldots\).

Chapter 5: Taylor series and approximations

5.1. (a) For \(f(x) = e^{\frac{1}{2}x}\),

\[
f'(x) = \frac{1}{2}e^{\frac{1}{2}x}, \quad f''(x) = \frac{1}{4}e^{\frac{1}{2}x}, \quad f'''(x) = \frac{1}{8}e^{\frac{1}{2}x},
\]
so that

\[
f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{4}, \quad f'''(0) = \frac{1}{8}.
\]

The Taylor polynomial approximation to four terms becomes

\[
f(x) \approx 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3.
\]

We can estimate that the three term approximation will be accurate to two decimal places if for the fourth term

\[
\left| \frac{1}{8.3!}x^3 \right| < 0.005, \text{ or } |x| < 0.621\ldots.
\]

(b) For \(f(x) = (1 + x)^{\frac{1}{2}}\), the Taylor approximation is

\[
(1 + x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.
\]

The three-term approximation will be accurate to two decimal places if

\[
\left| \frac{x^3}{16} \right| < 0.005 \text{ or } |x| < 0.432\ldots.
\]

(c) For \(f(x) = (1 + x)^{-\frac{1}{4}}\), the four-term Taylor polynomial is

\[
(1 + x)^{-\frac{1}{4}} \approx 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4.
\]

The three-term approximation will be accurate to two decimal places if

\[
\frac{14}{81}|x|^3 < 0.005 \text{ or } |x| < 0.306\ldots.
\]

(d) The Taylor approximation to four terms for \(\sin 2x\) can be obtained form the series for \(\sin y\) where \(y = 2x\) (use (5.4c)):

\[
\sin 2x \approx (2x) - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7
\]

\[
\approx 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7
\]

The three-term approximation will be accurate to two decimal places if

\[
\frac{8}{15}|x|^7 < 0.005 \text{ or } |x| < 0.196\ldots.
\]

(e) Using the expansion for \(\cos z\), where \(z = \frac{1}{2}x\) (see (5.4d)):

\[
\cos \frac{1}{2}x \approx 1 - \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6.
\]
The three-term polynomial will be accurate to two decimal places if \( \frac{1}{16} x^6 < 0.005 \) or \( |x| < 2.475 \ldots \)

(f) The four-term expansion is (see (5.4e)):
\[
\ln(1 + x) \approx x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4.
\]
The three-term polynomial will be accurate to two decimal places if \( \frac{1}{4} x^4 < 0.005 \) or \( |x| < 0.376 \ldots \)

(g) Let \( f(x) = (1 + x^2)^{\frac{1}{2}} \). Put \( u = x^2 \). Then, as in (b),
\[
(1 + x^2)^{\frac{1}{2}} = (1 + u)^{\frac{1}{2}} \approx 1 + \frac{1}{2} u - \frac{1}{8} u^2 + \frac{1}{16} u^3
\approx 1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \frac{1}{16} x^6.
\]
The three-term polynomial will be accurate to two decimal places if \( \frac{1}{16} x^6 < 0.005 \) or \( |x| < 0.656 \ldots \)

(h) The four-term Taylor polynomial for \( \ln(1 + 3x) \) is (put \( u = 3x \), etc.),
\[
\ln(1 + 3x) \approx (3x) - \frac{1}{2} (3x)^2 + \frac{1}{3} (3x)^3 - \frac{1}{4} (3x)^4
\approx 3x - \frac{9}{2} x^2 + 9x^3 - \frac{81}{4} x^4.
\]
The three-term approximation will be accurate to two decimal places if \( \frac{81}{4} x^4 < 0.005 \) or \( |x| < 0.125 \ldots \)

5.2. The Taylor expansion for \( f(x) \) about \( x = 0 \) is
\[
f(x) = f(0) + f'(0) x + \frac{1}{2!} f''(0) x^2 + \frac{1}{3!} f'''(0) x^3 + \cdots.
\]

(a) Let \( f(x) = e^x \). Then \( f'(x) = f''(x) = \cdots = e^x \). Hence
\[
f(0) = f'(0) = f''(0) = \cdots = 1.
\]

(b) Let \( f(x) = \sin x \). Then
\[
f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x, \text{ etc.}
\]
so that
\[
f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = 0, \ldots,
\]
the Taylor coefficients being
\[
1, \quad \frac{1}{1!}, \quad \frac{1}{2!}, \ldots.
\]

(c) Let \( f(x) = \cos x \). Then
\[
f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x, \ldots,
\]
so that
\[ f(0) = 1, \quad f'(0) = 0 \quad f''(0) = -1 \quad f'''(0) = 0, \quad f^{(4)}(0) = 1, \ldots. \]

(d) Let \( f(x) = (1 + x)^\alpha \). Then
\[ f'(x) = \alpha(1 + x)^{\alpha - 1}, \quad f''(x) = \alpha(\alpha - 1)(1 - x)^{\alpha - 2}, \ldots. \]
so that
\[ f(0) = 1, \quad f'(0) = \alpha, \quad f''(0) = \alpha(\alpha - 1), \ldots. \]

(e) Let \( f(x) = \ln(1 + x) \). Then
\[ f'(x) = 1/(1 + x), \quad f''(x) = -1/(1 + x)^2, \quad f'''(x) = -2 \times 1/(1 + x)^3, \quad f^{(4)}(x) = 3 \times 2 \times 1/(1 + x)^4, \ldots. \]
so that
\[ f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2!, \quad f^{(4)}(0) = 3!, \ldots. \]

Therefore the coefficient of \( x^n \) for \( n \geq 1 \) is
\[ (-1)^{n-1}(n-1)! \quad n! = (-1)^{n-1} \frac{n}{n}. \]

5.3. (a) The general term for \( e^x \) is \( x^n/n! \). Hence, for four-decimal point accuracy we require \( n \) such that, for \(|x| = 2\),
\[ \frac{x^n}{n!} = \frac{2^n}{n!} < 0.00005. \]
For \( n = 11, \frac{2^n}{n!} = 0.000051 \ldots \) and for \( n = 12, \frac{2^n}{n!} = 0.0000085 \ldots < 0.00005 \). Hence terms up to \( x^{11} \) are required.

(b) The general term for \( \sin x \) is \( (-1)^n x^{2n-1}/(2n-1)! \). Hence for four-decimal accuracy we require the smallest \( n \) such that, for \(|x| = 2\),
\[ \left| \frac{x^{2n-1}}{(2n-1)!} \right| = \frac{2^{2n-1}}{(2n-1)!} < 0.00005. \]
For \( n = 6, \frac{2^{2n-1}}{(2n-1)!} = 0.000051 \ldots \) and for \( n = 7, \frac{2^{2n-1}}{(2n-1)!} = 0.0000031 \ldots \) . Hence terms up to and including \( x^{11} \) are required.

(c) The general term for \( \cos x \) is \( (-1)^n x^{2n}/(2n)! \). Hence for four decimal accuracy we require the smallest \( n \) such that, for \(|x| = 2\),
\[ \left| \frac{x^{2n}}{(2n)!} \right| = \frac{2^{2n}}{(2n)!} < 0.00005. \]
For \( n = 5, \frac{2^{2n}}{(2n)!} = 0.000028 \ldots \) and for \( n = 6, \frac{2^{2n}}{(2n)!} = 0.0000085 \ldots \) . Hence terms up to and including \( x^{10} \) are required.

(d) For \((1 + x)^\frac{1}{2}\) the general term is
\[ \frac{\alpha(\alpha - 1)(\alpha - 2)\ldots(\alpha - n + 1)}{n!} x^n, \]
where \( \alpha = -\frac{1}{2} \). For four-decimal accuracy we require the smallest \( n \) such that, for \(|x| = 0.5\),
\[ \left| \frac{\alpha(\alpha - 1)(\alpha - 2)\ldots(\alpha - n + 1)}{n!} (0.5)^n \right| < 0.00005. \]
For \( n = 8 \) the last term has magnitude 0.000051 \ldots, and for \( n = 9 \) the magnitude is 0.000021 \ldots.
(e) For \( \ln(1 + x) \), the general term in its Taylor series is \((-1)^{n+1}x^n/n\). For four-decimal accuracy we require the smallest \( n \) such that, for \( |x| = 0.5 \),
\[
\left| (-1)^{n+1}x^n/n \right| = 0.5^n/n < 0.00005.
\]
For \( n = 10, 0.5^n/n = 0.00097 \ldots \), whilst for \( n = 11, 0.5^n/n = 0.000044 \ldots \). Hence terms up to and including \( x^{10} \) are required.

5.4. (a) Let \( f(x) = \arcsin x \). Then
\[
\begin{align*}
f'(x) & = \frac{1}{\sqrt{1 - x^2}}, \quad f''(x) = \frac{x}{(1 - x^2)^{3/2}}, \quad f'''(x) = \frac{1 + 2x^2}{(1 - x^2)^{5/2}}. \\
\end{align*}
\]
so that \( f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1 \). Hence
\[
\arcsin x = x + \frac{1}{6}x^3 + \ldots.
\]
(b) Let \( f(x) = \arccos x \). Then
\[
\begin{align*}
f'(x) & = -\frac{1}{\sqrt{1 - x^2}}, \quad f''(x) = -\frac{x}{(1 - x^2)^{3/2}}, \\
\end{align*}
\]
so that \( f(0) = \frac{1}{2}\pi, f'(0) = -1, f''(0) = 0 \). The Taylor series starts with
\[
\arccos x = \frac{1}{2}\pi - x + \ldots.
\]
(c) Let \( f(x) = \arctan x \). Then
\[
\begin{align*}
f'(x) & = \frac{1}{1 + x^2}, \quad f''(x) = -\frac{2x}{(1 + x^2)^2}, \quad f'''(x) = -\frac{2 + 6x^2}{(1 + x^2)^3}. \\
\end{align*}
\]
Hence \( f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2 \). The Taylor series starts with
\[
\arctan x = x - \frac{1}{3}x^3 + \ldots.
\]
(d) Let \( f(x) = e^{-x}\sin x \). Then
\[
\begin{align*}
f'(x) & = e^{-x}(\cos x - \sin x), \quad f''(x) = -2e^{-x}\cos x. \\
\end{align*}
\]
Hence \( f(0) = 0, f'(0) = 1, f''(0) = -2 \). The Taylor series starts with
\[
e^{-x}\sin x = x - x^2 + \ldots.
\]
(e) Let \( f(x) = e^{-x}\cos x \). Then
\[
\begin{align*}
f'(x) & = -e^{-x}(\cos x + \sin x). \\
\end{align*}
\]
Hence \( f(0) = 1, f'(0) = -1 \) so that the Taylor series starts with
\[
e^{-x}\cos x = 1 - x.
\]
5.5. (a) Let \( f(x) = 1/(1 + 3x) \). Then \( f'(x) = -3/(1 + 3x)^2, f''(x) = 18/(1 + 3x)^3 \) so that
\[
f(0) = 1, \quad f'(0) = -3, \quad f''(0) = 18.
\]
The first three terms of its Taylor series are
\[
\frac{1}{1+3x} = 1 - 3x + 9x^2 + \cdots.
\]
Alternatively, the binomial expansion (5.4f) can be used. Also from (5.4f) the expansion will be valid for
\[-1 < 3x < 1 \text{ or } -\frac{1}{3} < x < \frac{1}{3}.\]
(b) Adapting (5.4f),
\[
\frac{1}{2-x} = 2^{-1} \left[ 1 + \left( -\frac{x}{2} \right) \right]^{-1} = \frac{1}{2} \left[ 1 + (-1) \left( -\frac{x}{2} \right) + \left( \frac{-1)(-2)}{2!} \right) \left( -\frac{x}{2} \right)^2 + \cdots \right]
\]
the series is valid for $-1 < \frac{1}{2}x < 1$ or $-2 < x < 2$.
(c) Using (5.4f) again
\[
(3-x)^{\frac{1}{2}} = 3^{\frac{1}{2}} \left( 1 - \frac{1}{3}x \right) = 3^{\frac{1}{2}} \left[ 1 - \frac{1}{9}x - \frac{1}{81}x^2 - \cdots \right].
\]
The series is valid for $-3 < x < 3$.
(d) Using (5.4f) again
\[
(x-3)^{\frac{1}{2}} = (-3)^{\frac{1}{2}} \left[ 1 + \left( -\frac{x}{3} \right) \right]^{\frac{1}{2}} = (3)^{\frac{1}{2}} \left[ 1 - \frac{1}{9}x + \frac{1}{81}x^2 + \cdots \right].
\]
The series is valid for $-3 < x < 3$.
(e) Adapting (5.4e),
\[
\ln(9-x) = \ln \left[ 9 \left( 1 - \frac{1}{9}x \right) \right] = \ln 9 + \ln \left( 1 - \frac{1}{9}x \right) = 2 \ln 3 - \frac{1}{9}x - \frac{1}{162}x^2 - \cdots.
\]
The series is valid for $-9 < x < 9$.
(f) From (5.4d),
\[
\cos \left( \frac{1}{2}x \right) = 1 - \frac{1}{2!} \left( \frac{x}{2} \right)^2 + \frac{1}{4!} \left( \frac{x}{2} \right)^4 - \cdots = 1 - \frac{1}{8}x^2 + \frac{1}{384}x^4 - \cdots.
\]
The series is valid for all $x$.
(g) Put $u = x^{\frac{1}{2}}$ for $x > 0$, and use (5.4c):
\[
\sin (x^{\frac{1}{2}}) = \sin u = u - \frac{1}{6}u^3 + \frac{1}{120}u^5 - \cdots = x^{\frac{1}{2}} - \frac{1}{6}x^{\frac{3}{2}} + \frac{1}{120}x^{\frac{5}{2}} - \cdots.
\]
Since $x^{\frac{1}{2}}$ is not real for $x < 0$ the series will be valid only for $x \geq 0$.
(h) Put $u = x^{\frac{1}{2}}$ for $x > 0$, and use (5.4d):
\[
\cos (x^{\frac{1}{2}}) = 1 - \frac{1}{2}x + \frac{1}{24}x^2 + \cdots.
\]
The series is valid for all $x \geq 0$. 

50
5.6. Multiply standard expansions for

\[ e^{-x} = 1 - x + \frac{1}{2} x^2 + \cdots \] \quad \text{and} \quad \frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 + \cdots. \]

Hence

\[ \frac{e^{-x}}{1+x} = \left(1 - x + \frac{1}{2} x^2 + \cdots\right) \left(1 - x + x^2 + \cdots\right), \]

\[ = 1 - 2x + \frac{5}{2} x^2 + \cdots. \]

(b) As in (a)

\[ (1-x)^{\frac{1}{2}}e^x = \left(1 - \frac{1}{2} x - \frac{1}{8} x^2 - \cdots\right) \left(1 + \frac{1}{2} x^2 + \cdots\right), \]

\[ = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \cdots. \]

(c) This time square the series for \(\ln(1-x)\):

\[ \frac{1}{x^2} [\ln(1-x)]^2 = \frac{1}{x^2} \left[-x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \cdots\right]^2 \]

\[ = \frac{1}{x^2} \left[x^2 + x^3 + \frac{11}{12} x^4 + \cdots\right] \]

\[ = 1 + x + \frac{11}{12} x^2 + \cdots. \]

5.7. Start with the Taylor series

\[ 1 + \ln(1 + x) = 1 + x - \frac{1}{2} x^2 + \cdots. \]

Then, assuming that the expansion takes the form \(b_0 + b_1 x + b_2 x^2 + \cdots\),

\[ \frac{1}{1 + \ln(1 + x)} = \frac{1}{1 + x - \frac{1}{2} x^2 + \cdots} = b_0 + b_1 x + b_2 x^2 + \cdots. \]

We now equate coefficients of powers of \(x\) in the identity

\[ 1 = \left(1 + x - \frac{1}{2} x^2 + \cdots\right) \left(b_0 + b_1 x + b_2 x^2 + \cdots\right), \]

\[ = b_0 + (b_1 + b_0) x + (b_2 - b_1 - \frac{1}{2} b_0) x^2 + \cdots \]

Hence \(b_0 = 1, b_1 = -b_0 = -1, b_2 = b_1 + \frac{1}{2} b_0 = \frac{3}{2}\) and the Taylor series starts with

\[ \frac{1}{[1 + \ln(1 + x)]} = 1 - x + \frac{3}{2} x^2 + \cdots. \]

(b) Write \(\tan x = \sin x / \cos x\) and use the series for \(\sin x\) and \(\cos x\). Thus

\[ \tan x = \frac{x - \frac{1}{3} x^3 + \frac{1}{30} x^5 + \cdots}{1 - x + \frac{1}{6} x^3 + \cdots} \]

\[ = b_1 x + b_3 x^3 + b_5 x^5 + \cdots. \]
Note that the series will contain only odd powers of \( x \). By cross-multiplying
\[
x - \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots = \left( 1 - \frac{1}{2} x + \frac{1}{6} x^3 + \cdots \right) \left( b_1 x + b_3 x^3 + b_5 x^5 + \cdots \right).
\]
By matching the coefficients of \( x, x^2, \ldots \) on either side we obtain
\[
b_1 = 1, \quad b_3 = \frac{1}{3}, \quad b_5 = \frac{2}{15},
\]
so that
\[
\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \cdots.
\]
(c) From (5.4b)
\[
1 + e^x = 2 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \cdots.
\]
Assume that
\[
\frac{1}{1 + e^x} = b_0 + b_1 x + b_2 x^2 + \cdots;
\]
then
\[
1 = \left( 2 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \cdots \right) \left( b_0 + b_1 x + b_2 x^2 + b_3 x^3 \cdots \right)
\]
\[
= 2b_0 + (b_0 + 2b_1) x + \left( \frac{1}{2} b_0 + b_1 + 2b_2 \right) x^2 + \cdots.
\]
Solving for \( b_0, b_1, b_2 \) and \( b_3 \),
\[
\frac{1}{1 + e^x} = \frac{1}{2} - \frac{1}{4} x + \frac{1}{48} x^3 + \cdots.
\]
(d) Use the definition
\[
tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},
\]
where
\[
\sinh x = x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots, \quad \cosh x = 1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots.
\]
Hence, if the required series is \( b_1 x + b_3 x^3 + b_5 x^5 + \cdots \) (it must be an odd function), then
\[
x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots = \left( 1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots \right) \left( b_1 x + b_3 x^3 + b_5 x^5 + \cdots \right)
\]
\[
= x + (b_1 x + (b_3 + \frac{1}{2} b_1) x^3 + (b_5 + \frac{1}{2} b_3 + \frac{1}{24} b_1) x^5 + \cdots
\]
Comparing powers of \( x \), it follows that \( b_1 = 1, b_3 = -\frac{1}{3}, b_5 = \frac{2}{15} \) so that
\[
tanh x = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 + \cdots.
\]
(e) Since \( x/\sin x \) is an even function and \( \sin x = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 \), then
\[
x = \left( x - \frac{1}{6} x^3 + \frac{1}{120} x^5 \right) \left( b_0 + b_2 x^2 + b_4 x^4 + \cdots \right)
\]
\[
= b_0 x + \left( b_2 - \frac{1}{6} b_0 \right) + \left( b_4 - \frac{1}{6} b_2 + \frac{1}{120} b_0 \right) x^4 + \cdots.
\]
Hence \( b_0 = 1, b_2 = \frac{1}{6} \) and \( b_4 = \frac{7}{360} \) so that
\[
\frac{x}{\sin x} = 1 + \frac{1}{6} x^2 + \frac{7}{360} x^4 + \cdots.
\]
5.8. The following series provide approximations for large values of \( x \).

(a) Let \( u = 1/x \). Then

\[
\left( 1 - \frac{1}{x} \right)^{\frac{1}{2}} = (1 - u)^{\frac{1}{2}} = 1 - \frac{1}{2}u - \frac{1}{8}u^2 + \cdots \quad \text{(binomial series)}
\]

which will be valid for \(|u| < 1\), or equivalently, \(|x| > 1\).

(b) Let \( x > 0 \) and \( u = 1/x^{\frac{1}{2}} > 0 \). Then

\[
\ln \left( 1 + \frac{1}{x^{\frac{1}{2}}} \right) = \ln(1 + u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + \cdots
\]

This series will be valid for \( 0 < u \leq 1 \), or \( x \geq 1 \).

(c) Let \( x > 0 \) and \( u = 1/x \). Then

\[
\frac{x^{\frac{1}{2}}}{(1 + x)^{\frac{1}{2}}} = \frac{1}{(1 + \frac{1}{x})^{\frac{1}{2}}} = (1 + u)^{-1}
\]

\[
= 1 - \frac{1}{2}u + \frac{3}{8}u^2 + \cdots \quad \text{(binomial expansion)}
\]

The series is valid for \( 0 < u < 1 \) or \( x > 1 \).

(d) Let \( u = (1/x) + (1/x^2) \). Then, using (5.4e),

\[
\ln(1 + x + x^2) = \ln(x^2) + \ln(1 + u) = \ln(x^2) + u - \frac{1}{2}u^2 + \cdots
\]

\[
= \ln(x^2) + \left( \frac{1}{x} + \frac{1}{x^2} \right) - \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x^2} \right)^2 + \cdots
\]

The series is valid for \(-1 < (1/x)+(1/x^2) < 1\), that is, when \( x < \frac{1}{2}(\sqrt{5}-1) \) or when \( x > \frac{1}{2}(1+\sqrt{5}) \).

(e) Let \( u = 1/x \). Then

\[
\frac{1}{\sin(1/x)} = \frac{1}{\sin u} = \frac{1}{u - \frac{1}{6}u^3 + \frac{1}{720}u^5 + \cdots}
\]

\[
= \frac{1}{u} \cdot \frac{1}{1 - \frac{1}{6}u^2 + \frac{1}{720}u^4 + \cdots}
\]

\[
= \frac{1}{u} \left[ 1 + \left( \frac{1}{6}u^2 + \frac{1}{120}u^4 \right) + \left( \frac{1}{6}u^2 + \frac{1}{120}u^4 \right)^2 + \cdots \right]
\]

\[
= \frac{1}{u} \left[ 1 + \frac{1}{6}u^2 + \frac{7}{360}u^4 + \cdots \right]
\]

\[
= x + \frac{1}{6x} + \frac{7}{360x^3} + \cdots
\]

5.9. (a) Using the two-term Taylor expansion for \( \sin x \),

\[
\frac{1}{\sin x} \approx \frac{1}{x - \frac{1}{6}x^3} \approx \frac{1}{x} \left( 1 - \frac{1}{6}x^2 \right)
\]
\[
\frac{1}{x} \left(1 + \frac{1}{6}x^2\right) \quad \text{(using (5.4f))}
\]
\[
\approx \frac{1}{x} + \frac{1}{6}x,
\]
for small \(x\). Note that for \(x = 0.5\), the error is
\[
\left| \sin x - \left(\frac{1}{x} + \frac{1}{6}x\right) \right| = 0.0024 \ldots,
\]
that is, about 0.1%.

(b) Write as
\[
(1 + x)^{\frac{1}{2}} = x^{\frac{1}{2}} \left(1 + \frac{1}{x}\right)^{\frac{1}{2}} \approx x^{\frac{1}{2}} \left(1 + \frac{1}{2x}\right)
\]
for large \(x\).

(c) Using the result from (b)
\[
(2 + x)^{\frac{1}{2}} - (1 + x)^{\frac{1}{2}} = 2^{\frac{1}{2}}(1 + \frac{1}{2}x)^{\frac{1}{2}} - (1 + x)^{\frac{1}{2}}
\approx 2^{\frac{1}{2}} \left(\frac{x}{2}\right)^{\frac{1}{2}} + (2x)^{\frac{1}{2}} - \left[x^{\frac{1}{2}} + \frac{1}{2x^{\frac{1}{2}}}\right]
\approx \frac{1}{2x^{\frac{1}{2}}},
\]
for large \(x\).

(d) Using the three-term Taylor expansion for \(\cos x\),
\[
\frac{1}{(1 - \cos x)^{\frac{1}{2}}} \approx \left[\frac{x^2}{2} - \frac{x^4}{24}\right]^{-\frac{1}{2}}
\]
\[
\approx \frac{2^{\frac{1}{2}}}{x} \left[1 + \frac{1}{24}x^2\right] \quad \text{(binomial expansion)}
\]
\[
\approx \frac{2^{\frac{1}{2}}}{x} + \frac{2^{\frac{1}{2}}x}{24},
\]
for small \(x\).

5.10. (a) Expanding about \(x = 1\),
\[
\ln x = \ln[1 + (x - 1)] = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 + \cdots \quad \text{(using (5.4e))}.
\]
The series is valid for \(-1 < x - 1 < 1\) or \(0 < x < 2\).

(b) For an expansion about \(x = \frac{1}{2}\pi\), write
\[
\cos x = \cos\left[\frac{1}{2}\pi + (x - \frac{1}{2}\pi)\right] = -\sin(x - \frac{1}{2}\pi).
\]
Now use the Taylor expansion for the sine:
\[
\cos x = -\sin(x - \frac{1}{2}\pi) = -(x - \frac{1}{2}\pi) + \frac{1}{3!}(x - \frac{1}{2}\pi)^3 - \frac{1}{5!}(x - \frac{1}{2}\pi)^5 + \cdots
\]
The series is valid for all \(x - \frac{1}{2}\pi\), which means for all \(x\).
(c) For a Taylor series centred at \( x = 1 \), write

\[
(1 + x)^{\frac{1}{2}} = |2 + (x - 1)|^{\frac{1}{2}} = 2^{\frac{1}{2}}[1 + \frac{1}{2}(x - 1)]^{\frac{1}{2}}.
\]

Now expand the right-hand side using the binomial series (5.4f):

\[
(1 + x)^{\frac{1}{2}} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x - 1) - \frac{1}{16\sqrt{2}}(x - 1)^2 + \cdots.
\]

The series is valid for \(-1 < \frac{1}{2}(x - 1) < 1\), that is for \(-1 < x < 3\).

5.11. (a) Since \( f(x) \) has a stationary point at \( x = c \), \( f'(c) = 0 \) and its Taylor series about \( x = c \) will be

\[
f(x) = f(c) + \frac{1}{2}f''(c)x^2 + \frac{1}{3!}f'''(c)x^3 + \cdots.
\]

Approximately

\[
f(x) \approx f(c) + \frac{1}{2}f''(c)x^2,
\]

for \( |x - c| \) small. Hence if \( f''(c) > 0 \), then \( f(x) > f(c) \) close to \( x = c \) excluding \( x = c \). The conclusion is that \( x = c \) is a minimum. Similarly if \( f''(c) < 0 \), then \( x = c \) is a maximum.

(b) If \( f''(c) = 0 \), then we must look at the signs of higher derivatives. Suppose that \( f^{(N)}(c) \neq 0 \) is the first non-zero derivative (that is, \( f^{(r)}(c) = 0 \) for \( r = 1, 2, \ldots, N - 1 \)). Hence, approximately,

\[
f(x) \approx f(c) + \frac{1}{N!}f^{(N)}(c)(x - c)^N.
\]

If \( N \) is even and \( f^{(N)}(c) > 0 \) then \( x = c \) is a minimum, whilst if \( f^{(N)}(c) < 0 \) then \( x = c \) is a maximum. If \( N \) is odd then the stationary point will be a point of inflection.

5.12. (Compare eqn (2.15).) Put \( (e^x - 1)/x = f(x) \) for \( x \neq 0 \), and use the Taylor series (5.4b) for \( e^x \):

\[
f(x) = [(1 + x + \frac{1}{2!}x^2 + \cdots) - 1]/x = 1 + \frac{1}{2!}x + \cdots,
\]

for \( x \neq 0 \). Therefore \( \lim_{x \to 0} f(x) = 1 \), which is the ‘missing value’ at \( x = 0 \).

(b) Put \( (1 - \cos x)/x^2 = f(x) \) for \( x \neq 0 \). From (5.4d)

\[
f(x) = \frac{1}{x^2}[1 - (1 - \frac{1}{2!} + \frac{1}{4!}x^4 - \cdots)]
= \frac{1}{x^2}(\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots)
= \frac{1}{2!} - \frac{1}{4!}x^2 + \cdots \quad (\text{for } x \neq 0)
\]

Therefore \( \lim_{x \to 0} f(x) = \frac{1}{2}. \)

(c) Put \( \ln(1 + x) - x/\sin x = f(x), \ x \neq 0. \) From (5.4e),

\[
f(x) = \frac{(x - \frac{1}{2}x^2 + \cdots) - x}{x - \frac{1}{3!}x^3 + \cdots} = \frac{-\frac{1}{2}x^2 + \cdots}{x - \frac{1}{3!}x^3 + \cdots}
= \frac{x(-\frac{1}{2} + \cdots)}{1 - \frac{1}{3!}x^3 + \cdots} \quad (\text{for } x \neq 0).
\]

Therefore \( \lim_{x \to 0} f(x) = 0. \)

(Alternatively, rewrite \( f(x) \) in the form

\[
\frac{\ln(1 + x) - x}{x} \frac{x}{\sin x}
\]
and use the limits (2.13) and (2.14).)
(d) Put \( \frac{\sin x}{1 - \cos x} = f(x), \ x \neq 0 \):

\[
f(x) = \frac{x - \frac{1}{3} x^3 + \cdots}{1 - (1 - \frac{1}{3} x^2 + \cdots)} = \frac{x(1 - \frac{1}{3} x^2 + \cdots)}{\frac{1}{2} x^2(1 - \frac{1}{4} x^2 + \cdots)}.
\]

This does not tend to a limit; it approaches \( \infty \) as \( x \to 0 \). Therefore this function does not possess a fill-in value at \( x = 0 \) which would make it continuous.

5.13. (a)
\[
\lim_{x \to 0} \frac{(1 - x)^{12} - 1}{(1 - x)^{10} - 1} = \lim_{x \to 0} \frac{-12x + \text{higher powers}}{-10x + \text{higher powers}} = \frac{12}{10},
\]
where the binomial theorem (4.7) was used to expand the powers of \( (1 - x) \).

(b)
\[
\lim_{x \to 0} \frac{\sin x - x}{x - x \cos x} = \lim_{x \to 0} \frac{x - \frac{1}{6} x^3 + \cdots - x}{(x - \frac{1}{3} x^3 + \cdots - x(1 - \frac{1}{3} x^2 + \cdots))} = \lim_{x \to 0} \frac{-\frac{1}{6} x + \text{higher powers}}{\frac{1}{3} x + \text{higher powers}} = \lim_{x \to 0} \frac{-\frac{1}{6} x + \text{higher powers}}{\frac{1}{3} x + \text{higher powers}} = \frac{-1}{2}.
\]

(c) Put \( x = \pi + u \). Then
\[
\lim_{x \to \pi} \frac{\cos x + 1}{x - \pi} = \lim_{u \to 0} \frac{\cos(\pi + u) + 1}{u} = \lim_{u \to 0} \frac{\cos \pi \cos u - \sin \pi \sin u + 1}{u} = \lim_{u \to 0} \frac{-\cos u + 1}{u} = \lim_{u \to 0} \frac{(1 - \frac{1}{2} u^2 + \cdots) + 1}{u} = \lim_{u \to 0} u\left(\frac{1}{2} + \cdots\right) = 0.
\]

(d) Put \( x = u + \frac{1}{2} \pi \): then
\[
\lim_{x \to \frac{1}{2} \pi} \frac{\sin x - 1}{\cos 5x} = \lim_{u \to 0} \frac{\sin(\frac{1}{2} \pi + u) - 1}{\cos(\frac{1}{2} \pi + 5u)} = \lim_{u \to 0} \frac{\sin \frac{1}{2} \pi \cos u + \cos \frac{1}{2} \pi \sin u - 1}{\cos \frac{1}{2} \pi \cos 5u - \sin \frac{1}{2} \pi \sin 5u} = \lim_{u \to 0} \frac{\cos u - 1}{-\sin 5u} = \lim_{u \to 0} \frac{\frac{1}{2} u^2 - \cdots}{-5u + \cdots} = 0
\]

5.14. Let
\[
f(x) = \frac{e^x - 1}{e^x - 1 - x} = \frac{(e^x - 1)/x}{[(e^x - 1)/x] - 1}.
\]
Since \( \lim_{x \to 0} [(e^x - 1)/x] = 1 \) (see eqn (2.15) or Problem 5.12a), \( f(x) \) approaches infinity as \( x \to 0 \).
To determine the question of signs, return to the original form and take the following steps:

(i) \( e^x - 1 \) is negative when \( x < 0 \) and positive when \( x > 0 \).

(ii) \( \frac{d}{dx}(e^x - 1 - x) = e^x \), which is greater than zero for all \( x \). Therefore \( e^x - 1 - x \) is a steadily increasing function for all \( x \). Since also it is zero at \( x = 0 \), it must be negative when \( x < 0 \) and positive for \( x > 0 \).

(iii) From (i) and (ii), \( f(x) \) is negative when \( x < 0 \) and positive when \( x > 0 \). Therefore \( f(x) \to -\infty \) as \( x \to 0 \) from the left, and \( f(x) \to +\infty \) as \( x \to 0 \) from the right.

5.15. (a) 

\[
\lim_{x \to 0} \left[ \frac{\sin^3 3x}{1 - \cos x} \right] = \lim_{x \to 0} \frac{(3x + \cdots)^3}{1 - (1 - \frac{1}{2}x^2 + \cdots)} = \lim_{x \to 0} \frac{27x^3 + \cdots}{\frac{1}{2}x^2 + \cdots} = \lim_{x \to 0} \frac{27x(1 + \cdots)}{\frac{1}{2}(1 + \cdots)} = 0.
\]

(b) Let \( \left[ (e^x - 1)/x \right]^{1/2} = g(x) \). We know from eqn (2.15) that 

\[
\lim_{x \to 0} [g(x)]^2 = \lim_{x \to 0} \left[ \frac{e^x - 1}{x} \right] = 1.
\]

But also 

\[
\lim_{x \to 0} [g(x)]^2 = \lim_{x \to 0} [g(x)g(x)] = \lim_{x \to 0} g(x) \lim_{x \to 0} g(x) = [\lim_{x \to 0} g(x)]^2.
\]

Therefore 

\[
\lim_{x \to 0} g(x) = \sqrt{1} = 1,
\]

the positive square root being taken because \( g(x) \) is never negative.

(c) 

\[
\lim_{x \to 0} \left[ \frac{(2 + \tan x) \sin x}{x(3 - \tan^2 x)} \right] = \lim_{x \to 0} \left[ \frac{2 + \tan x}{3 - \tan^2 x} \right] \lim_{x \to 0} \left[ \frac{\sin x}{x} \right] = \frac{2 + 0}{3 - 0} = \frac{2}{3} \quad \text{(where we refer to eqn (2.13))}
\]

5.16. Let \( f(x) = 3x - \sin x \) and \( g(x) = x \). Then 

\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{3 - \cos x}{1} \bigg|_{x=0} = 2.
\]

5.17. In the following, \( S \) represents the required sum.

(a) 

\[
S = \sum_{n=0}^{\infty} \frac{1}{n!}x^n - \sum_{n=0}^{\infty} (-x)^n = e^x - (1 - x + x^2 - \cdots) = e^x - \frac{1}{1 + x},
\]

the second term being a geometric series with common ratio \( -x \): see (5.4a).

(b) \( S = x^3 + \frac{1}{2}x^4 + \frac{1}{3}x^5 + \cdots = x^2(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots) \). From (5.4e), 

\[
\ln(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots.
\]

Therefore \( S = -x^2 \ln(1 - x) \).
(c) \( S = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \). But from (5.4b),
\[
e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \text{ and } e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots.
\]
Therefore \( e^x + e^{-x} = 2S \), and
\[
S = \frac{1}{2}(e^x + e^{-x}) = \cosh x
\]
from (1.26).

5.18. In the following, \( S \) represents the required sum.
(a) From (5.4b)
\[
e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots,
\]
and
\[
e^{-2} = 1 - 2 + \frac{2^2}{2!} - \frac{2^3}{3!} + \cdots,
\]
so that
\[
e^2 - e^{-2} = 2 \left( 2 + \frac{2^2}{3!} + \frac{2^3}{5!} + \cdots \right),
\]
or
\[
S = \frac{1}{2}(e^2 - e^{-2}).
\]
(b) From (5.4b), \( S = e^\frac{1}{2} \).
(c) \( S \) is geometric with common ratio \((-\frac{1}{4})\). Therefore, by (5.4a),
\[
S = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5}.
\]

Chapter 6: Complex numbers

6.1. (a) \( x = -1 \pm i \); (b) \( x = 3 \pm i \); (c) \( x = i \) or \(-3i\).

6.2. The equation is a quadratic equation in \( x^2 \). Hence \( x^2 = -4 \) or \( 1 \). Taking square roots
\[
x = \pm 1 \text{ or } \pm 2i.
\]

6.3. The standard form of a complex number is \( a + ib \), where \( a \) and \( b \) are real numbers. Thus the answers are (a) \( 4 + 3i \); (b) \( 3 - 5i \); (c) \(-11 + 15i \); (d) \( 9 + 3i \); (e) \( \frac{1}{2} + \frac{1}{2}i \); (f) \( 1 + 6i \); (g) \(-\frac{1}{2} - \frac{1}{2}i \).

6.4. The boundary between real and complex roots in the \( (p, q) \) plane is the parabola \( p^2 = 4q \):
the roots are real if \( p^2 \geq 4q \) and complex if \( p^2 < 4q \). The roots are both real and negative in the quadrant \( p > 0, q < 0 \).

6.5. (a) \( 4 + i \); (b) \( 5 + 5i \); (c) \( \frac{1}{3} - \frac{2}{3}i \); (d) \( -\frac{11}{25} - \frac{9}{25}i \).

6.6. (a) \(-4i \); (b) \(-7 + 4i \); (c) \(-\frac{1}{2} + \frac{3}{2}i \); (d) \(-\frac{1}{5} - \frac{2}{5}i \).

6.7. (a) \( 1 - i \); (b) \( 2i \); (c) \(-2i \); (d) \( \frac{1}{2}(1 + gm) \); (e) \( i \).

6.8. Numerically to 3 decimal places the answers are: (a) \( 4.482 + 2.218i \); (b) \( 16.233 - 0.167i \); (c) \(-1.248 + 2.728i \); (d) \( 88.669 \); (e) \( 266.050 + 0.512i \).

6.9. (a) \( |z_1| = 2\sqrt{2}, \text{ Arg } z_1 = \frac{\pi}{4} \); (b) \( |z_2| = 8, \text{ Arg } z_2 = -\frac{\pi}{3} \); (c) \( |z_3| = 5, \text{ Arg } z_3 = -\frac{\pi}{2} \); (d) \( |z_4| = 3, \text{ Arg } z_4 = \pi \); (e) \( |z_5| = 5, \text{ Arg } z_5 = \arctan(\frac{4}{3}) \).

6.10. The curves are: (a) the circle \( x^2 + y^2 = 1 \); (b) the straight line \( y = 2 \); (c) The circle \( (x - a_1)^2 + (y - a_2)^2 = 1 \) where \( a_1 = \text{Re}(a) \) and \( a_2 = \text{Im}(a) \); (d) the parabola \( y^2 = 4x \); (e) the ellipse \( 3x^2 + 4y^2 = 12 \) (need to square twice to remove the square roots); the complex formula
expresses the well-known property of ellipses that the sum of the distances from any point on an ellipse to the foci is a constant; (f) the straight line \( y = x \) for \( x \geq 0 \); (g) the archimedean spiral \( r = \theta \).

6.11. (a) \( \sqrt{2} \exp(\frac{3}{4}i\pi) \); (b) \( 2 \exp(i\pi) \); (c) \( 3 \exp(-\frac{1}{2}i\pi) \); (d) \( 14 \exp(-\frac{3}{4}i\pi) \); (e) \( 2 \sqrt{2} \exp(i\theta) \) where \( \theta = \arctan[(\sqrt{3} - 1)/(\sqrt{3} + 1)] \); (f) \( \frac{\sqrt{2}}{1+\sqrt{3}} \exp(-\frac{1}{4}i\pi) \); (g) \( 2 \exp(i) \); (h) \( \sqrt{2} \exp(i) \) where \( \theta = \arctan[\cos/(\cos + \sin)] \); (i) \( 512 \exp(i\pi) \); (j) \( \sqrt{2} \exp(\frac{3}{4}i\pi) \).

6.12. Use the identity
\[
\exp(i(\theta_1 + \theta_2)) = \exp(i\theta_1) \exp(i\theta_2).
\]

Hence
\[
\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)
= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)
= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2).
\]

Equating real and imaginary parts it follows that
\[
\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2,
\]
and
\[
\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2.
\]

For the other identities use
\[
\exp(i(\theta_1 - \theta_2)) = \exp(i\theta_1) \exp(-i\theta_2).
\]

6.13.

![Figure 58: Problem: 6.13](image)

6.14 For the general case with \( f(\theta) = a \cos \theta + b \sin \theta \),
\[
f'(\theta) = -a \sin \theta + b \cos \theta,
\]
and
\[
f''(\theta) = -a \cos \theta - b \sin \theta = -f(\theta).
\]
The first case can be obtained by putting \( a = 1 \) and \( b = i \).

6.15. Using exponential forms for \( \cos \) and \( \sin \), it follows that
\[
\tan a = \frac{\sin ia}{\cos ia} = \frac{2[\exp(ai^2) - \exp(-ai^2)]}{2i[\exp(ai^2) + \exp(-ai^2)]} = \frac{\exp(-a) - \exp(a)}{i \cdot \exp(-a) + \exp(a)} = i \exp(a) - \exp(-a) = i \exp(a) + \exp(-a) = i \tan h a.
\]
6.16. (a) The equation \( \cosh z = 1 \) implies 
\[
\frac{1}{2}(e^z + e^{-z}) = 1 \Rightarrow e^{2z} - 2e^z + 1 = 0 \Rightarrow (e^z - 1)^2 = 0.
\]
Hence \( e^z = 1 \). If \( z = a + bi \) (a, b real), then \( e^{a+bi} = 1 = e^{2n\pi i} \), \( n = 0, \pm 1, \pm 2, \ldots \). Thus \( a = 0 \) and \( b = 2n\pi \). The roots are given by \( z = 2n\pi i \), \( n = 0, \pm 1, \pm 2, \ldots \).

(b) Similarly \( \sinh z = 1 \) implies 
\[
e^{2z} - 2e^z - 1 = 0.
\]
Hence \( e^z = 1 \pm \sqrt{2} \). If \( z = a + bi \), then 
\[
e^{a+bi} = e^a(\cos b + i\sin b) = 1 \pm \sqrt{2}.
\]
It follows that \( \sin b = 0 \) so that \( b = n\pi, \ n = 0, \pm 1, \pm 2, \ldots \). Hence 
\[
e^a \cos b = e^a \cos(n\pi) = (-1)^n e^a = 1 \pm \sqrt{2},
\]
so that 
\[
a = \ln(\sqrt{2} - 1), \ (n \text{ odd}) \quad a = \ln(\sqrt{2} + 1) \ (n \text{ even}).
\]
The complex roots are 
\[
z = \ln(\sqrt{2} - 1) + in\pi, \ (n \text{ odd}) \quad z = \ln(\sqrt{2} + 1) + in\pi \ (n \text{ even}).
\]
(c) \( e^z = -1 = e^{2n+1}i \), \( n = \ldots -2, -1, 0, 1, 2, \ldots \). It follows that the roots are 
\[
z = (2n + 1)\pi i \quad (n = \ldots -2, -1, 0, 1, 2, \ldots).
\]
(d) \( \cos z = \sqrt{2} \) implies that 
\[
\frac{1}{2}(e^{iz} + e^{-iz}) = \sqrt{2} \Rightarrow e^{2iz} - 2\sqrt{2}e^{iz} + 1 = 0.
\]
Hence 
\[
e^{iz} = \sqrt{2} \pm 1 = (\sqrt{2} \pm 1)e^{2n\pi i}, \quad (n = 0, \pm 1, \pm 2, \ldots).
\]
If \( z = a + ib \), then 
\[
e^{-b} = (\sqrt{2} \pm 1), \quad \text{and} \quad a = 2n\pi.
\]
Hence the roots are 
\[
z = 2n\pi - i\ln(\sqrt{2} \pm 1).
\]

6.17. (a) \( \log(1 + i\sqrt{3}) = \ln(\sqrt{1 + 3}) + i\arg(1 + i\sqrt{3}) = \ln 2 + \frac{1}{2}i\pi. \)
(b) We can write \( \log z = \log|z| + i(\arg z + 2k\pi) \) where \( k \) is an integer. Hence if \( \log z = \pi i \), then \( |z| = 1 \) and \( \arg z + 2k\pi = \pi \) so that \( k = 0 \) and \( z = \pi i \) is the only solution.
(c) \( \log(ei) = \ln(e) + \frac{1}{2}i\pi = 1 + \frac{1}{2}i\pi. \)
(d) \( e^{\log z} = e^{\ln r + i\theta + 2k\pi i} = e^{\ln r} e^{i\theta} = re^{i\theta} = z. \) Therefore \( \log z \) defines the set of functions inverse to \( e^z \), as is suggested by the notation.

6.18. (a) \( 2^i = e^{i\ln 2} = \cos(\ln 2) + i\sin(\ln 2). \)
(b) 
\[
i = e^{i\ln i} = \exp[i \ln(e^{\frac{1}{2}\pi i})] = e^{\frac{1}{2}i\pi}.
\]
This number is real: hence \( \arg(i) = 0. \)
(c) The equation becomes 
\[
2^i = e^{i\ln 2} = e^{i\log|z| + i(\arg z + 2k\pi)} = e^{-(\arg z + 2k\pi)} e^{\log|z|} = -1 = e^{(2n+1)i\pi},
\]
where \( n \) and \( k \) are integers. Hence 
\[
\log|z| = (2n + 1)\pi, \quad \arg z = 0, \quad k = 0.
\]
Therefore, the solutions are given by \( z = e^{(2n+1)\pi i} \), where \( n \) is any integer.

6.19. Write the equation as
\[
z^5 = -1 = e^{(2n+1)\pi i}, \quad n, \text{ any integer}.
\]
The solutions are given by
\[
z = \exp \left[ \frac{1}{5} (2n + 1) \pi i \right], \quad (n = 1, 2, 3, 4, 5).
\]
Other values of \( n \) merely repeat these solutions. On the Argand diagram the solutions all lie on the unit circle centred at the origin at polar angles \( \frac{1}{5} \pi \), \( \frac{3}{5} \pi \), \( \pi \), \( \frac{7}{5} \pi \), \( \frac{9}{5} \pi \).

6.20. Denote the complex number by \( z \) in each case.
(a) \( z = 2 e^{3+2i} = 2 e^3 [\cos 2 + i \sin 2] \). Hence
\[
|z| = 2e^3, \quad \text{Arg} \ z = 2, \quad \text{Re} \ z = 2e^3 \cos 2, \quad \text{Im} \ z = 2e^3 \sin 2.
\]
(b) \( z = 4 e^i = 4[\cos 1 + i \sin 1] \). Hence
\[
|z| = 4, \quad \text{Arg} \ z = 1, \quad \text{Re} \ z = 4 \cos 1, \quad \text{Im} \ z = 4 \sin 1.
\]
(c) \( z = 5 \exp[\cos(\frac{1}{4} \pi) + i \sin(\frac{1}{4} \pi)] = 5 \exp(1/\sqrt{2})[\cos(1/\sqrt{2}) + i \sin(1/\sqrt{2})] \). Hence
\[
|z| = 5 \exp(1/\sqrt{2}), \quad \text{Arg} \ z = 1/\sqrt{2},
\]
\[
\text{Re} \ z = 5 \exp(1/\sqrt{2}) \cos(1/\sqrt{2}), \quad \text{Im} \ z = 5 \exp(1/\sqrt{2} \sin(1/\sqrt{2})).
\]
(d) \( z = e^{1+i} = e(\cos 1 + i \sin 1) \). Hence
\[
|z| = e, \quad \text{Arg} \ z = 1, \quad \text{Re} \ z = e \cos 1, \quad \text{Im} \ z = e \sin 1.
\]

6.21. Let \( z = ce^{\alpha+i\beta} = ce^\alpha[\cos \beta + i \sin \beta] \). Comparing with
\[
x = 0.04e^{-0.01t} \sin 12t,
\]
we can identify
\[
ce = 0.04, \quad \alpha = -0.01t, \quad \beta = 12t + \frac{1}{2} \pi.
\]

6.22. We have to express the sine as a cosine. Hence
\[
i(t) = ce^{-0.05t} \sin(0.4t + 0.5) = ce^{-0.05t} \cos(0.4t + 0.5 - \frac{1}{2} \pi),
\]
\[
= \text{Re}[ce^{-0.05t} e^{i(0.4t+0.5-\frac{1}{4} \pi)}],
\]
\[
= \text{Re}[ce^{-0.05t+i(0.4t+0.5-\frac{1}{4} \pi)}]
\]

61
6.23. (a) \( z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy \). Hence
\[
\text{Re}(z^2) = x^2 - y^2, \quad \text{Im}(z^2) = 2xy.
\]
(b) First
\[
z^3 = (x + iy)(x^2 - y^2 + 2ixy) = (x^3 - 3xy^2) + i(3x^2y - y^3).
\]
Thus
\[
z + 2z^2 + 3z^3 = (x + 2x^2 - 2y^2 + 3x^3 - 9xy^2) + i(y + 4xy + 9x^2y - 3y^3).
\]
Hence
\[
\text{Re}(z + 2z^2 + 3z^3) = x + 2x^2 - 2y^2 + 3x^3 - 9xy^2,
\]
\[
\text{Im}(z + 2z^2 + 3z^3) = y + 4xy + 9x^2y - 3y^3.
\]
(c) \( \sin z = \sin(x + iy) = \frac{1}{2i} [e^{i(x+iy)} - e^{-i(x+iy)}] \)
\[
= \frac{1}{2i} [e^{-y}e^{ix} - e^y e^{-ix}]
\]
\[
= \frac{1}{2i} [e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)]
\]
\[
= e^{-y} \sin x.
\]
Hence
\[
\text{Re}(\sin z) = e^{-y} \sin x, \quad \text{Im}(\sin z) = 0.
\]
(d) \( \cos z = \frac{1}{2} [e^{i(x+iy)} + e^{-i(x+iy)}] = \frac{1}{2} [e^{ix}e^{-y} + e^{-ix}e^y] \)
\[
= \cos x \cosh y - i \sin x \sinh y.
\]
Hence
\[
\text{Re}(\cos z) = \cos x \cosh y, \quad \text{Im}(\cos z) = -\sin x \sinh y.
\]
(e) Using (d)
\[
e^z \cos z = \frac{1}{2} e^{x+iy} [\cos x \cosh y - i \sin x \sinh y]
\]
\[
= \frac{1}{2} e^x (\cos y + i \sin y) [\cos x \cosh y - i \sin x \sinh y]
\]
\[
= \frac{1}{2} e^x [(\cos y \cos x \cosh y + \sin y \sin x \sinh y) + (\sin y \cos x \cosh y - \cos y \sin x \sinh y)].
\]
(f) \( \exp(z^2) = \exp(x^2 - y^2) \exp(2xyi) = \exp(x^2 - y^2) [\cos 2xy + i \sin 2xy]. \) Hence
\[
\text{Re}[\exp(z^2)] = \exp(x^2 - y^2) \cos 2xy, \quad \text{Im}[\exp(z^2)] = \exp(x^2 - y^2) \sin 2xy.
\]
6.24. \( w = u + iv = f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi. \) Hence, equating real and imaginary parts
\[
u = x^2 - y^2, \quad v = 2xy.
\]
The hyperbolas map into the straight lines \( u = 1 \) and \( v = 2 \) respectively in the \( w \) plane.
6.25. Substituting for \( z \) it follows that
\[
w = z + \frac{c}{z} = x + iy + \frac{c(x - iy)}{x^2 + y^2}
\]
\[
= \left(x + \frac{cx}{x^2 + y^2}\right) + i \left(y - \frac{cy}{x^2 + y^2}\right).
\]
For the circle $|z| = 1$, $x^2 + y^2 = 1$, so that

$$w = x(1 + c) + iy((1 - c)$$

Hence $u = x(1 + c)$ and $v = y(1 - c)$. Thus, on the circle $|z| = 1$

$$x^2 + y^2 = 1 = \frac{u^2}{(1 + c)^2} + \frac{v^2}{(1 - c)^2},$$

which is the equation of an ellipse.

6.26. The derivation of the formula for $\cos^6 \theta$ is given in Example 6.20. For $\sin^6 \theta$ use the identity

$$\sin n\theta = \frac{1}{2i} \left(z^n - \frac{1}{z^n}\right),$$

where $z = \cos \theta + i \sin \theta$. Then

$$
(2 \sin \theta)^6 = - \left(z - \frac{1}{z}\right)^6
= - \left(z^6 + \frac{1}{z^6}\right) + 6 \left(z^4 + \frac{1}{z^4}\right) - 15 \left(z^2 + \frac{1}{z^2}\right) + 20
= 2 (- \cos 6\theta + 6 \cos 4\theta - 15 \cos 2\theta + 20).
$$

Finally

$$\sin^6 \theta = \frac{1}{32} (- \cos 6\theta + 6 \cos 4\theta - 15 \cos 2\theta + 20).$$

6.27. The displacement is given by

$$x = \text{Re} \ z = e^{-0.2t} \cos 0.5t.$$ 

Hence $x = 0$ where $\cos 0.5t = 0$. The required zeros of $x$ are given by

$$0.5t = \frac{1}{2} (2n + 1)\pi, \quad \text{for integer } n.$$

Hence $t = (2n + 1)\pi, \ (n = 0, \pm 1, \pm 2, \ldots)$. 

The velocity is given by

$$\frac{dx}{dt} = \frac{d}{dx} [e^{-0.2t} \cos 0.5t] = -e^{-0.2t} [0.2 \cos 0.5t + 0.5 \sin 0.5t],$$

or, alternatively, by

$$\text{Re} \frac{dz}{dt} = \text{Re} \left[ \frac{d}{dt} e^{(-0.2 + 0.5i)t} \right]
= \text{Re} \left[ (-0.2 + 0.5i)e^{-0.2t} (\cos 0.5t + i \sin 0.5t) \right]
= -e^{-0.2t}[0.2 \cos 0.5t + 0.5 \sin 0.5t].$$

6.28. If $z = 2 + i$ is a solution then so is its conjugate $2 - i$ since the coefficients of the polynomial are real. Therefore $(z - 2 - i)(z - 2 + i) = z^2 - 4z + 5$ is a factor. Hence

$$z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 - 4z + 5)(z^2 + 2z + 2).$$

The other solutions are given by $z^2 + 2z + 2 = 0$, that is, $z = -1 \pm i$.

6.29. (a)

$$S = 1 - \sin \theta + \frac{1}{2!} \sin 2\theta - \frac{1}{3!} \sin \theta + \cdots
= \text{Im} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n e^{i\theta}}{n!} \right]
= \text{Im} \left[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \right],$$

63
where $z = -e^{i\theta}$. The infinite series is the Taylor series for the exponential function $e^z$ (see Section 5.4). Hence

$$S = 1 + \text{Im}[e^z] = 1 + \text{Im}[\exp(-e^{i\theta})]$$

$$= 1 + \text{Im}[\exp(-\cos \theta - i \sin \theta)]$$

$$= 1 - e^{-\cos \theta} \sin(\sin \theta).$$

(b) In this case

$$T = 1 + 2 \cos \theta + \frac{2^2}{2!} \cos 2\theta + \frac{2^3}{3!} \cos 3\theta + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} \cos n\theta = \text{Re} \left[ \sum_{n=0}^{\infty} \frac{2^n}{n!} e^{in\theta} \right]$$

Using the Taylor series for an exponential function;

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} e^{in\theta} = \sum_{n=0}^{\infty} \frac{(2e^{i\theta})^n}{n!} = \exp[2e^{i\theta}] = \exp[2\cos \theta + 2i \sin \theta]$$

$$= e^{2 \cos \theta} \left[ \cos(2 \sin \theta) + i \sin(2 \sin \theta) \right]$$

Hence

$$T = e^{2 \cos \theta} \cos(2 \sin \theta).$$