Chapter 11
Models using time series data

11.1 Overview

This chapter introduces the application of regression analysis to time series data, beginning with static models and then proceeding to dynamic models with lagged variables used as explanatory variables. It is shown that multicollinearity is likely to be a problem in models with unrestricted lag structures and that this provides an incentive to use a parsimonious lag structure, such as the Koyck distribution. Two models using the Koyck distribution, the adaptive expectations model and the partial adjustment model, are described, together with well-known applications to aggregate consumption theory, Friedman’s permanent income hypothesis in the case of the former and Brown’s habit persistence consumption function in the case of the latter. The chapter concludes with a discussion of prediction and stability tests in time series models.

11.2 Learning outcomes

After working through the corresponding chapter in the text, studying the corresponding slideshows, and doing the starred exercises in the text and the additional exercises in this subject guide, you should be able to:

- explain why multicollinearity is a common problem in time series models, especially dynamic ones with lagged explanatory variables
- describe the properties of a model with a lagged dependent variable (ADL(1,0) model)
- describe the assumptions underlying the adaptive expectations and partial adjustment models
- explain the properties of OLS estimators of the parameters of ADL(1,0) models
- explain how predetermined variables may be used as instruments in the fitting of models using time series data
- explain in general terms the objectives of time series analysts and those constructing VAR models.
### 11. Models using time series data

#### 11.3 Additional exercises

A11.1 The output below shows the result of linear and logarithmic regressions of expenditure on food on income, relative price, and population (measured in thousands) using the Demand Functions data set, together with the correlations among the variables. Provide an interpretation of the regression coefficients and perform appropriate statistical tests.

```
Dependent Variable: FOOD
Method: Least Squares
Sample: 1959 2003
Included observations: 45

=================================================================================================
Variable Coefficient Std. Error t-Statistic Prob.
===============================================
  C  -19.49285  88.86914 -0.219343  0.8275
  DPI   0.031713  0.010658  2.975401  0.0049
  PRELFOOD  0.403356  0.365133  1.104681  0.2757
  POP    0.001140  0.000563  2.024017  0.0495

R-squared 0.988529  Mean dependent var 422.0374
Adjusted R-squared 0.987690  S.D. dependent var 91.58053
S.E. of regression 10.16104  Akaike info criteri7.559685
Sum squared resid 4233.113  Schwarz criterion 7.720278
Log likelihood -166.0929  F-statistic 1177.745
Durbin-Watson stat 0.404076  Prob(F-statistic) 0.000000
```

```
Dependent Variable: LGFOOD
Method: Least Squares
Sample: 1959 2003
Included observations: 45

=================================================================================================
Variable Coefficient Std. Error t-Statistic Prob.
===============================================
  C    5.293654  2.766987  1.916077  0.0623
  LGDPI  0.589239  0.080158  7.351014  0.0000
  LGPRFOOD  -0.122598  0.084355 -1.453361  0.1537
  LGPOP    -0.289219  0.258762 -1.117706  0.2702

R-squared 0.992245  Mean dependent var 6.021331
Adjusted R-squared 0.991678  S.D. dependent var 0.222787
S.E. of regression 0.020324  Akaike info criteri-4.869317
Sum squared resid 0.016936  Schwarz criterion -4.708725
Log likelihood 113.5596  F-statistic 1748.637
Durbin-Watson stat 0.488502  Prob(F-statistic) 0.000000
```

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11.3. Additional exercises

A11.2 Perform regressions parallel to those in Exercise A11.1 using your category of expenditure and provide an interpretation of the coefficients.

A11.3 The output shows the result of a logarithmic regression of expenditure on food per capita, on income per capita, both measured in US$ million, and the relative price index for food. Provide an interpretation of the coefficients, demonstrate that the specification is a restricted version of the logarithmic regression in Exercise A11.1, and perform an $F$ test of the restriction.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-5.425877</td>
<td>0.353655</td>
<td>-15.34231</td>
<td>0.0000</td>
</tr>
<tr>
<td>LGDPIPC</td>
<td>0.280229</td>
<td>0.014641</td>
<td>19.14024</td>
<td>0.0000</td>
</tr>
<tr>
<td>LGPRFOOD</td>
<td>0.052952</td>
<td>0.082588</td>
<td>0.641160</td>
<td>0.5249</td>
</tr>
</tbody>
</table>

A11.4 Perform a regression parallel to that in Exercise A11.3 using your category of expenditure. Provide an interpretation of the coefficients, and perform an $F$ test of the restriction.

A11.5 The output shows the result of a logarithmic regression of expenditure on food per capita, on income per capita, the relative price index for food, and population. Provide an interpretation of the coefficients, demonstrate that the specification is equivalent to that for the logarithmic regression in Exercise A11.1, and use it to perform a $t$ test of the restriction in Exercise A11.3.
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Included observations: 45

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>5.293654</td>
<td>2.762757</td>
<td>1.916077</td>
<td>0.0623</td>
</tr>
<tr>
<td>LGDPIPC</td>
<td>0.589239</td>
<td>0.080158</td>
<td>7.351014</td>
<td>0.0000</td>
</tr>
<tr>
<td>LGPRFOOD</td>
<td>-0.122598</td>
<td>0.084355</td>
<td>-1.453361</td>
<td>0.1537</td>
</tr>
<tr>
<td>LGPOP</td>
<td>-0.699980</td>
<td>0.179299</td>
<td>-3.903973</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

R-squared 0.947037  Mean dependent var -6.321984
Adjusted R-squared 0.943161  S.D. dependent var 0.085249
S.E. of regression 0.020324  Akaike info criter -4.869317
Sum squared resid 0.016936  Schwarz criterion -4.708725
Log likelihood 113.5596  F-statistic 244.3727
Durbin-Watson stat 0.488502  Prob(F-statistic) 0.000000

---

A11.6 Perform a regression parallel to that in Exercise A11.5 using your category of expenditure, and perform a $t$ test of the restriction implicit in the specification in Exercise A11.4.

A11.7 In Exercise 11.9 you fitted the model:

$$LGCAT = \beta_1 + \beta_2 LGDPI + \beta_3 LGDPI(-1) + \beta_4 LGPRCAT + \beta_5 LGPRCAT(-1) + u$$

where $CAT$ stands for your category of expenditure.

- Show that $(\beta_2 + \beta_3)$ and $(\beta_4 + \beta_5)$ are theoretically the long-run (equilibrium) income and price elasticities.
- Reparameterise the model and fit it to obtain direct estimates of these long-run elasticities and their standard errors.
- Confirm that the estimates are equal to the sum of the individual short-run elasticities found in Exercise 11.9.
- Compare the standard errors with those found in Exercise 11.9 and state your conclusions.

A11.8 In a certain bond market, the demand for bonds, $B_t$, in period $t$ is negatively related to the expected interest rate, $i_{t+1}^e$, in period $t + 1$:

$$B_t = \beta_1 + \beta_2 i_{t+1}^e + u_t \quad (1)$$

where $u_t$ is a disturbance term not subject to autocorrelation. The expected interest rate is determined by an adaptive expectations process:

$$i_{t+1}^e - i_t^e = \lambda (i_t - i_t^e) \quad (2)$$

where $i_t$ is the actual rate of interest in period $t$. A researcher uses the following model to fit the relationship:

$$B_t = \gamma_1 + \gamma_2 i_t + \gamma_3 B_{t-1} + v_t \quad (3)$$

where $v_t$ is a disturbance term.
11.3. Additional exercises

- Show how this model may be derived from the demand function and the adaptive expectations process.
- Explain why inconsistent estimates of the parameters will be obtained if equation (3) is fitted using ordinary least squares (OLS). (A mathematical proof is not required. Do not attempt to derive expressions for the bias.)
- Describe a method for fitting the model that would yield consistent estimates.
- Suppose that $u_t$ was subject to the first-order autoregressive process:
  $$u_t = \rho u_{t-1} + \epsilon_t$$
  where $\epsilon_t$ is not subject to autocorrelation. How would this affect your answer to the second part of this question?
- Suppose that the true relationship was actually:
  $$B_t = \beta_1 + \beta_2 i_t + u_t \quad (1*)$$
  with $u_t$ not subject to autocorrelation, and the model is fitted by regressing $B_t$ on $i_t$ and $B_{t-1}$, as in equation (3), using OLS. How would this affect the regression results?
- How plausible do you think an adaptive expectations process is for modelling expectations in a bond market?

A11.9 The output shows the result of a logarithmic regression of expenditure on food on income, relative price, population, and lagged expenditure on food using the Demand Functions data set. Provide an interpretation of the regression coefficients, paying attention to both short-run and long-run dynamics, and perform appropriate statistical tests.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.487655</td>
<td>2.072156</td>
<td>0.717921</td>
<td>0.4771</td>
</tr>
<tr>
<td>LGDPI</td>
<td>0.143829</td>
<td>0.090334</td>
<td>1.592194</td>
<td>0.1194</td>
</tr>
<tr>
<td>LGPRFOOD</td>
<td>-0.095749</td>
<td>0.061118</td>
<td>-1.566613</td>
<td>0.1253</td>
</tr>
<tr>
<td>LGPOP</td>
<td>-0.046515</td>
<td>0.189453</td>
<td>-0.245524</td>
<td>0.8073</td>
</tr>
<tr>
<td>LGFOOD(-1)</td>
<td>0.727290</td>
<td>0.113831</td>
<td>6.389195</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

R-squared = 0.995886, Mean dependent var = 6.030691
Adjusted R-squared = 0.995464, S.D. dependent var = 0.216227
S.E. of regression = 0.014564, Akaike info criter = -5.513937
Sum squared resid = 0.008272, Schwarz criterion = -5.311888
Log likelihood = 126.3066, F-statistic = 2359.938
Durbin-Watson stat = 1.103102, Prob(F-statistic) = 0.000000

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A11.10 Perform a regression parallel to that in Exercise A11.9 using your category of expenditure. Provide an interpretation of the coefficients, and perform appropriate statistical tests.

A11.11 In his classic study *Distributed Lags and Investment Analysis* (1954), Koyck investigated the relationship between investment in railcars and the volume of freight carried on the US railroads using data for the period 1884–1939. Assuming that the desired stock of railcars in year $t$ depended on the volume of freight in year $t - 1$ and year $t - 2$ and a time trend, and assuming that investment in railcars was subject to a partial adjustment process, he fitted the following regression equation using OLS (standard errors and constant term not reported):

$$
\hat{I}_t = 0.077F_{t-1} + 0.017F_{t-2} - 0.0033t - 0.110K_{t-1} \quad R^2 = 0.85
$$

where $I_t = K_t - K_{t-1}$ is investment in railcars in year $t$ (thousands), $K_t$ is the stock of railcars at the end of year $t$ (thousands), and $F_t$ is the volume of freight handled in year $t$ (ton-miles).

Provide an interpretation of the equation and describe the dynamic process implied by it. (Note: It is best to substitute $K_t - K_{t-1}$ for $I_t$ in the regression and treat it as a dynamic relationship determining $K_t$.)

A11.12 Two researchers agree that a model consists of the following relationships:

$$
Y_t = \alpha_1 + \alpha_2 X_t + u_t \quad (1)
$$

$$
X_t = \beta_1 + \beta_2 Y_{t-1} + v_t \quad (2)
$$

$$
Z_t = \gamma_1 + \gamma_2 Y_t + \gamma_3 X_t + \gamma_4 Q_t + w_t \quad (3)
$$

where $u_t$, $v_t$, and $w_t$, are disturbance terms that are drawn from fixed distributions with zero mean. It may be assumed that they are distributed independently of $Q_t$ and of each other and that they are not subject to autocorrelation. All the parameters may be assumed to be positive and it may be assumed that $\alpha_2\beta_2 < 1$.

- One researcher asserts that consistent estimates will be obtained if (2) is fitted using OLS and (1) is fitted using IV, with $Y_{t-1}$ as an instrument for $X_t$. Determine whether this is true.

- The other researcher asserts that consistent estimates will be obtained if both (1) and (2) are fitted using OLS, and that the estimate of $\beta_2$ will be more efficient than that obtained using IV. Determine whether this is true.
11.4 Answers to the starred exercises in the textbook

The table gives the data used by Cobb and Douglas (1928) to fit the original Cobb–Douglas production function:

\[ Y_t = \beta_1 K_t^{\beta_2} L_t^{\beta_3} v_t \]

\( Y_t, K_t, \) and \( L_t, \) being index number series for real output, real capital input, and real labour input, respectively, for the manufacturing sector of the United States for the period 1899–1922 (1899 = 100). The model was linearised by taking logarithms of both sides and the following regressions was run (standard errors in parentheses):

\[
\hat{\log} Y = -0.18 + 0.23 \log K + 0.81 \log L \quad R^2 = 0.96 \\
(0.43) \quad (0.06) \quad (0.15)
\]

Provide an interpretation of the regression coefficients.

**Answer:**

The elasticities of output with respect to capital and labour are 0.23 and 0.81, respectively, both coefficients being significantly different from zero at very high significance levels. The fact that the sum of the elasticities is close to one suggests that there may be constant returns to scale. Regressing output per worker on capital per worker, one has:

\[
\hat{\log} \frac{Y}{L} = 0.01 + 0.25 \log \frac{K}{L} \quad R^2 = 0.63 \\
(0.02) \quad (0.04)
\]

The smaller standard error of the slope coefficient suggests a gain in efficiency.

Fitting a reparameterised version of the unrestricted model:

\[
\hat{\log} \frac{Y}{L} = -0.18 + 0.23 \log \frac{K}{L} + 0.04 \log L \quad R^2 = 0.64 \\
(0.43) \quad (0.06) \quad (0.09)
\]

we find that the restriction is not rejected.
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11.7 The Cobb–Douglas model in Exercise 11.6 makes no allowance for the possibility that output may be increasing as a consequence of technical progress, independently of $K$ and $L$. Technical progress is difficult to quantify and a common way of allowing for it in a model is to include an exponential time trend:

$$Y_t = \beta_1 K_t^{\beta_2} L_t^{\beta_3} e^{\rho t} v_t$$

where $\rho$ is the rate of technical progress and $t$ is a time trend defined to be 1 in the first year, 2 in the second, etc. The correlations between log $K$, log $L$ and $t$ are shown in the table. Comment on the regression results.

$$\hat{\log} Y = 2.81 - 0.53 \log K + 0.91 \log L + 0.047 t \quad R^2 = 0.97$$

<table>
<thead>
<tr>
<th></th>
<th>LGK</th>
<th>LGL</th>
<th>TIME</th>
</tr>
</thead>
<tbody>
<tr>
<td>LGK</td>
<td>1.000000</td>
<td>0.909562</td>
<td>0.996834</td>
</tr>
<tr>
<td>LGL</td>
<td>0.909562</td>
<td>1.000000</td>
<td>0.896344</td>
</tr>
<tr>
<td>TIME</td>
<td>0.996834</td>
<td>0.896344</td>
<td>1.000000</td>
</tr>
</tbody>
</table>

**Answer:**

The elasticity of output with respect to labour is higher than before, now implausibly high given that, under constant returns to scale, it should measure the share of wages in output. The elasticity with respect to capital is negative and nonsensical. The coefficient of time indicates an annual exponential growth rate of 4.7 per cent, holding $K$ and $L$ constant. This is unrealistically high for the period in question. The implausibility of the results, especially those relating to capital and time (correlation 0.997), may be attributed to multicollinearity.

11.16 Demonstrate that the dynamic process (11.18) implies the long-run relationship given by (11.15).

**Answer:**

Equations (11.15) and (11.18) are:

$$\hat{Y} = \frac{\beta_1}{1 - \beta_3} + \frac{\beta_2}{1 - \beta_3} \hat{X}$$

$$Y_t = \beta_1 (1 + \beta_3 + \beta_3^2 + \cdots) + \beta_2 X_t + \beta_2 \beta_3 X_{t-1} + \beta_2 \beta_3^2 X_{t-2} + \cdots$$

Putting $X = \hat{X}$ for all $X$ in (11.18), and ignoring the disturbance terms, the long-run relationship between $Y$ and $X$ is given by:

$$\hat{Y} = \beta_1 (1 + \beta_3 + \beta_3^2 + \cdots) + \beta_2 \hat{X} + \beta_2 \beta_3 \hat{X} + \beta_2 \beta_3^2 \hat{X} + \cdots$$

$$= \frac{\beta_1}{1 - \beta_3} + (1 + \beta_3 + \beta_3^2 + \cdots) \beta_2 \hat{X}$$

$$= \frac{\beta_1}{1 - \beta_3} + \frac{\beta_2}{1 - \beta_3} \hat{X}.$$
11.17 The compound disturbance term in the adaptive expectations model (11.37) does potentially give rise to a problem that will be discussed in Chapter 12 when we come to the topic of autocorrelation. It can be sidestepped by representing the model in the alternative form.

\[ Y_t = \beta_1 + \beta_2 \lambda X_t + \beta_2 \lambda (1 - \lambda) X_{t-1} + \cdots + \beta_2 \lambda (1 - \lambda)^s X_{t-s} + \beta_2 (1 - \lambda)^{s+1} X_{t-s} + u_t. \]

Show how this form might be obtained, and discuss how it might be fitted.

**Answer:**

We start by reprising equations (11.31) – (11.34) in the text. We assume that the dependent variable \( Y_t \) is related to \( X^e_{t+1} \), the value of \( X \) anticipated in the next time period:

\[ Y_t = \gamma_1 + \gamma_2 X^e_{t+1} + u_t. \]  

(11.31)

To make the model operational, we hypothesise that expectations are updated in response to the discrepancy between what had been anticipated for the current time period, \( X^e_{t+1} \), and the actual outcome, \( X_t \):

\[ X^e_{t+1} - X^e_t = \lambda (X_t - X^e_t) \]  

(11.32)

where \( \lambda \) may be interpreted as a speed of adjustment. We can rewrite this as (11.33):

\[ X^e_{t+1} = \lambda X_t + (1 - \lambda) X^e_t. \]  

(11.33)

Hence we obtain (11.34):

\[ Y_t = \gamma_1 + \gamma_2 \lambda X_t + \gamma_2 (1 - \lambda) X^e_t + u_t. \]  

(11.34)

This includes the unobservable \( X^e_t \) on the right side. However, lagging (11.33), we have:

\[ X^e_t = \lambda X_{t-1} + (1 - \lambda) X^e_{t-1}. \]

Hence:

\[ Y_t = \gamma_1 + \gamma_2 \lambda X_t + \gamma_2 \lambda (1 - \lambda) X_{t-1} + \gamma_2 (1 - \lambda)^2 X^e_{t-1} + u_t. \]

This includes the unobservable \( X^e_{t-1} \) on the right side. However, continuing to lag and substitute, we have:

\[ Y_t = \gamma_1 + \gamma_2 \lambda X_t + \gamma_2 \lambda (1 - \lambda) X_{t-1} + \cdots + \gamma_2 \lambda (1 - \lambda)^s X_{t-s} + \gamma_2 (1 - \lambda)^{s+1} X^e_{t-s} + u_t. \]

Provided that \( s \) is large enough for \( \gamma_2 (1 - \lambda)^{s+1} \) to be very small, this may be fitted, omitting the unobservable final term, with negligible omitted variable bias.

We would fit it with a nonlinear regression technique that respected the constraints implicit in the theoretical structure of the coefficients.

11.19 The output below shows the result of fitting the model:

\[
LGFOOD = \beta_1 + \beta_2 LGDPI + \beta_2 \lambda (1 - \lambda) LGDPI(-1) + \beta_2 \lambda (1 - \lambda)^2 LGDPI(-2) \\
+ \beta_2 \lambda (1 - \lambda)^3 LGDPI(-3) + \beta_3 LGPRFOOD + u
\]

using the data on expenditure on food in the Demand Functions data set. \( LGFOOD \) and \( LGPRFOOD \) are the logarithms of expenditure on food and the
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relative price index series for food. C(1), C(2), C(3), and C(4) are estimates of $\beta_1$, $\beta_2$, $\lambda$ and $\beta_3$, respectively. Explain how the regression equation could be interpreted as an adaptive expectations model and discuss the dynamics implicit in it, both short-run and long-run. Should the specification have included further lagged values of LGDPI?

Answer:

Suppose that the model is:

$$ LGFOOD_t = \gamma_1 + \gamma_2 LGDPI_{t+1} + \gamma_3 LGPRFOOD_t + u_t $$

where $LGDPI_{t+1}$ is expected $LGDPI$ at time $t + 1$, and that expectations for income are subject to the adaptive expectations process:

$$ LGDPI_{t+1} - LGDPI_t = \lambda (LGDPI_t - LGDPI_t^e). $$

The adaptive expectations process may be rewritten:

$$ LGDPI_{t+1}^e = \lambda LGDPI_t + (1 - \lambda) LGDPI_t^e. $$

Lagging this equation one period and substituting, one has:

$$ LGDPI_{t+1}^e = \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + (1 - \lambda)^2 LGDPI_{t-2}. $$

Lagging a second time and substituting, one has:

$$ LGDPI_{t+1}^e = \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + \lambda(1-\lambda)^2 LGDPI_{t-2} + (1 - \lambda)^3 LGDPI_{t-3}. $$

Lagging a third time and substituting, one has:

$$ LGDPI_{t+1}^e = \lambda LGDPI_t + \lambda(1-\lambda) LGDPI_{t-1} + \lambda(1-\lambda)^2 LGDPI_{t-2} + \lambda(1-\lambda)^3 LGDPI_{t-3} + (1 - \lambda)^4 LGDPI_{t-4}. $$
Substituting this into the model, dropping the final unobservable term, one has the regression specification as stated in the question.

The estimates imply that the short-run income elasticity is 0.50. The speed of adjustment of expectations is 0.92. Hence the long-run income elasticity is $0.50/0.92 = 0.54$. The price side of the model has been assumed to be static. The estimate of the price elasticity is $-0.09$. The coefficient of the dropped unobservable term is $\gamma_2(1-\lambda)^4$. Given our estimates of $\gamma_2$ and $\lambda$, its estimate is 0.0003. Hence we are justified in neglecting it.

11.22 A researcher is fitting the following supply and demand model for a certain commodity, using a sample of time series observations:

\[
\begin{align*}
Q_{dt} &= \beta_1 + \beta_2 P_t + u_{dt} \\
Q_{st} &= \alpha_1 + \alpha_2 P_t + u_{st}
\end{align*}
\]

where $Q_{dt}$ is the amount demanded at time $t$, $Q_{st}$ is the amount supplied, $P_t$ is the market clearing price, and $u_{dt}$ and $u_{st}$ are disturbance terms that are not necessarily independent of each other. It may be assumed that the market clears and so $Q_{dt} = Q_{st}$.

- What can be said about the identification of (a) the demand equation, (b) the supply equation?
- What difference would it make if supply at time $t$ was determined instead by price at time $t-1$? That is:

\[
Q_{st} = \alpha_1 + \alpha_2 P_{t-1} + u_{st}.
\]

- What difference would it make if it could be assumed that $u_{dt}$ is distributed independently of $u_{st}$?

**Answer:**

The reduced form equation for $P_t$ is:

\[
P_t = \frac{1}{\alpha_2 - \beta_2} (\beta_1 - \alpha_1 + u_{dt} - u_{st}).
\]

$P_t$ is not independent of the disturbance term in either equation and so OLS would yield inconsistent estimates. There is no instrument available, so both equations are underidentified.

Provided that $u_{dt}$ is not subject to autocorrelation, $P_{t-1}$ could be used as an instrument in the demand equation. Provided that $u_{st}$ is not subject to autocorrelation, OLS could be used to fit the second equation. It makes no difference whether or not $u_{dt}$ is distributed independently of $u_{st}$.

The first equation could, alternatively, be fitted using OLS, with the variables switched. From the second equation, $P_{t-1}$ determines $Q_t$, and then, given $Q_t$, the demand equation determines $P_t$:

\[
P_t = \frac{1}{\beta_2} (Q_t - \beta_1 - u_{dt}).
\]

The reciprocal of the slope coefficient provides a consistent estimator of $\beta_2$. 

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11.24 Consider the following simple macroeconomic model:

\[ C_t = \beta_1 + \beta_2 Y_t + u_{C_t} \]
\[ I_t = \alpha_1 + \alpha_2 (Y_t - Y_{t-1}) + u_{I_t} \]
\[ Y_t = C_t + I_t \]

where \( C_t, I_t, \) and \( Y_t \) are aggregate consumption, investment, and income and \( u_{C_t} \) and \( u_{I_t} \) are disturbance terms. The first relationship is a conventional consumption function. The second relates investment to the change of output from the previous year. (This is known as an ‘accelerator’ model.) The third is an income identity. What can be said about the identification of the relationships in the model?

**Answer:**

The restriction on the coefficients of \( Y_t \) and \( Y_{t-1} \) in the investment equation complicates matters. A simple way of handling it is to define:

\[ \Delta Y_t = Y_t - Y_{t-1} \]

and to rewrite the investment equation as:

\[ I_t = \alpha_1 + \alpha_2 \Delta Y_t + u_{I_t}. \]

We now have four endogenous variables and four equations, and one exogenous variable. The consumption and investment equations are exactly identified. We would fit them using \( Y_{t-1} \) as an instrument for \( Y_t \) and \( \Delta Y_t \), respectively. The other two equations are identities and do not need to be fitted.

11.5 Answers to the additional exercises

A11.1 The linear regression indicates that expenditure on food increases by $0.032 billion for every extra $ billion of disposable personal income (in other words, by 3.2 cents out of the marginal dollar), that it increases by $0.403 billion for every point increase in the price index, and that it increases by $0.001 billion for every additional thousand population. The income coefficient is significant at the 1 per cent level (ignoring problems to be discussed in Chapter 12). The positive price coefficient makes no sense (remember that the dependent variable is measured in real terms). The intercept has no plausible interpretation.

The logarithmic regression indicates that the income elasticity is 0.59 and highly significant, and the price elasticity is -0.12, not significant. The negative elasticity for population is not plausible. One would expect expenditure on food to increase in line with population, controlling for other factors, and hence, as a first approximation, the elasticity should be equal to 1. However, an increase in population, keeping income constant, would lead to a reduction in income per capita and hence to a negative income effect. Given that the income elasticity is less than 1, one would still expect a positive elasticity overall for population. At least the estimate is not significantly different from zero. In view of the high correlation, 0.995, between \( LGDPI \) and \( LGPOP \), the negative estimate may well be a result of multicollinearity.
11.5. Answers to the additional exercises

### A11.2

#### OLS logarithmic regressions

<table>
<thead>
<tr>
<th></th>
<th>LGDPI</th>
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<th>LGP</th>
<th></th>
<th>LGPOP</th>
<th></th>
<th>R²</th>
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<td>coef.</td>
<td>s.e.</td>
<td>coef.</td>
<td>s.e.</td>
<td></td>
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<td>0.61</td>
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<tr>
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<td>0.28</td>
<td>−1.18</td>
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<td>2.79</td>
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<td>0.993</td>
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<td>0.26</td>
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<td>0.995</td>
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<td>0.26</td>
<td>0.13</td>
<td>−0.27</td>
<td>0.52</td>
<td>0.993</td>
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<tr>
<td>FLOW</td>
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<td>0.16</td>
<td>0.33</td>
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<td>1.21</td>
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<td>0.08</td>
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<td>0.982</td>
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<td>0.06</td>
<td>2.57</td>
<td>0.39</td>
<td>0.999</td>
</tr>
</tbody>
</table>

The price elasticities mostly lie in the range 0 to −1, as they should, and therefore seem plausible. However the very high correlation between income and population, 0.995, has given rise to a problem of multicollinearity and as a consequence the estimates of their elasticities are very erratic. Some of the income elasticities look plausible, but that may be pure chance, for many are unrealistically high, or negative when obviously they should be positive. The population elasticities are even less convincing.

### Correlations between prices, income and population

<table>
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<tr>
<th></th>
<th>LGP, LGDPI</th>
<th>LGP, LGPOP</th>
<th>LGP, LGDPI</th>
<th>LGP, LGPOP</th>
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</tbody>
</table>

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11. Models using time series data

A11.3 The regression indicates that the income elasticity is 0.40 and the price elasticity 0.21, the former very highly significant, the latter significant at the 1 per cent level using a one-sided test. If the specification is:

\[ \log \frac{FOOD}{POP} = \beta_1 + \beta_2 \log \frac{DPI}{POP} + \beta_3 \log PRELFOOD + u \]

it may be rewritten:

\[ \log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + (1 - \beta_2) \log POP + u. \]

This is a restricted form of the specification in Exercise A11.2:

\[ \log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + \beta_4 \log POP + u \]

with \( \beta_4 = 1 - \beta_2 \). We can test the restriction by comparing RSS for the two regressions:

\[ F(1, 41) = \frac{(0.023232 - 0.016936)/1}{0.016936/41} = 15.24. \]

The critical value of \( F(1, 40) \) at the 0.1 per cent level is 12.61. The critical value for \( F(1, 41) \) must be slightly lower. Thus we reject the restriction. Since the restricted version is misspecified, our interpretation of the coefficients of this regression and the \( t \) tests are invalidated.

A11.4 Given that the critical values of \( F(1, 41) \) at the 5 and 1 per cent levels are 4.08 and 7.31 respectively, the results of the \( F \) test may be summarised as follows:

- Restriction not rejected: CLOT, DENT, DOC, FURN, HOUS.
- Restriction rejected at the 5 per cent level: MAGS.
- Restriction rejected at the 1 per cent level: ADM, BOOK, BUSI, FLOW, FOOD, GAS, GASO, LEGL, MASS, OPHT, RELG, TELE, TOB, TOYS.

However, for reasons that will become apparent in the next chapter, these findings must be regarded as provisional.
### 11.5 Answers to the additional exercises

<table>
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<td>0.071656</td>
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<td>7.28</td>
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#### A11.5

If the specification is:

$$\log \frac{FOOD}{POP} = \beta_1 + \beta_2 \log \frac{DPI}{POP} + \beta_3 \log PRELFOOD + \gamma_1 POP + u$$

it may be rewritten:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + (1 - \beta_2 + \gamma_1) \log POP + u$$

This is equivalent to the specification in Exercise A11.1:

$$\log FOOD = \beta_1 + \beta_2 \log DPI + \beta_3 \log PRELFOOD + \beta_4 \log POP + u$$

with $\beta_4 = 1 - \beta_2 + \gamma_1$. Note that this is not a restriction. (1) – (3) are just different ways of writing the unrestricted model.

A $t$ test of $H_0: \gamma_1 = 0$ is equivalent to a $t$ test of $H_0: \beta_4 = 1 - \beta_2$, that is, that the restriction in Exercise A11.3 is valid. The $t$ statistic for $LGPOP$ in the regression is $-3.90$, and hence again we reject the restriction. Note that the test is equivalent to the $F$ test, $-3.90$ is the square root of $15.24$, the $F$ statistic, and it can be shown that the critical value of $t$ is the square root of the critical value of $F$.

#### A11.6

The $t$ statistics for all the categories of expenditure are supplied in the table in the answer to Exercise A11.4. Of course they are equal to the square root of the $F$ statistic, and their critical values are the square roots of the critical values of $F$, so the conclusions are identical and, like those of the $F$ test, should be treated as provisional.
11. Models using time series data

A11.7  *Show that* $\beta_2 + \beta_3$ *and* $(\beta_4 + \beta_5)$ *are theoretically the long-run (equilibrium) income and price elasticities.*

In equilibrium, $LGCAT = \overline{LGCAT}$, $LGDPI = \overline{LGDPI}(-1) = \overline{LGDPI}$ and $LGPRCAT = \overline{LGPRCAT}(-1) = \overline{LGPRCAT}$. Hence, ignoring the transient effect of the disturbance term:

\[
\begin{align*}
LGCAT &= \beta_1 + \beta_2 \overline{LGDPI} + \beta_3 \overline{LGDPI} + \beta_4 \overline{LGPRCAT} + \beta_5 \overline{LGPRCAT} \\
&= \beta_1 + (\beta_2 + \beta_3) \overline{LGDPI} + (\beta_2 + \beta_3) \overline{LGPRCAT}.
\end{align*}
\]

Thus the long-run equilibrium income and price elasticities are $\theta = \beta_2 + \beta_3$ and $\phi = \beta_4 + \beta_5$, respectively.

*Reparameterise the model and fit it to obtain direct estimates of these long-run elasticities and their standard errors.*

We will reparameterise the model to obtain direct estimates of $\theta$ and $\phi$ and their standard errors. Write $\beta_3 = \theta - \beta_2$ and $\phi = \beta_4 + \beta_5$ and substitute for $\beta_3$ and $\beta_5$ in the model. We obtain:

\[
\begin{align*}
LGCAT &= \beta_1 + \beta_2 \overline{LGDPI} + (\theta - \beta_2) \overline{LGDPI}(-1) + \beta_4 \overline{LGPRCAT} + (\theta - \beta_4) \overline{LGPRCAT}(-1) + u \\
&= \beta_1 + \beta_2 \overline{LGDPI} - \overline{LGDPI}(-1) + \theta \overline{LGDPI}(-1) + \beta_4 \overline{LGPRCAT} - \overline{LGPRCAT}(-1) + \phi \overline{LGPRCAT}(-1) + u \\
&= \beta_1 + \beta_2 \overline{LGDPI} + \theta \overline{LGDPI}(-1) + \beta_4 \overline{LGPRCAT} + \phi \overline{LGPRCAT}(-1) + u
\end{align*}
\]

where $DLGDPI = \overline{LGDPI} - \overline{LGDPI}(-1)$ and $DLGPRCAT = \overline{LGPRCAT} - \overline{LGPRCAT}(-1)$.

The output for $HOUS$ is shown below. $DLGPRCAT$ has been abbreviated as $DLGP$.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
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<tbody>
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R-squared: 0.999039  Mean dependent var: 6.379059
Adjusted R-squared: 0.998940  S.D. dependent var: 0.421861
S.E. of regression: 0.013735  Akaike info criterion: -5.531127
Sum squared resid: 0.007357  Schwarz criterion: -5.428379
Log likelihood: 128.8848  F-statistic: 10131.80
Durbin-Watson stat: 0.536957  Prob(F-statistic): 0.000000

*Confirm that the estimates are equal to the sum of the individual shortrun elasticities found in Exercise 11.9.*

The estimates of the long-run income and price elasticities are 1.01 and $-0.45$, respectively. The output below is for the model in its original form, where the
coefficients are all short-run elasticities. It may be seen that, for both income and price, the sum of the estimates of the short-run elasticities is indeed equal to the estimate of the long-run elasticity in the reparameterised specification.

Dependent Variable: LGHOUS
Method: Least Squares
Sample(adjusted): 1960 2003
Included observations: 44 after adjusting endpoints

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
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</thead>
<tbody>
<tr>
<td>C</td>
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<td>0.143844</td>
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<tr>
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R-squared 0.999039 Mean dependent var 6.379059
Adjusted R-squared 0.998940 S.D. dependent var 0.421861
S.E. of regression 0.013735 Akaike info criter-5.631127
Sum squared resid 0.007357 Schwarz criterion -5.428379
Log likelihood 128.8848 F-statistic 10131.80
Durbin-Watson stat 0.536957 Prob(F-statistic) 0.000000

Compare the standard errors with those found in Exercise 11.9 and state your conclusions.

The standard errors of the long-run elasticities in the reparameterised version are much smaller than those of the short-run elasticities in the original specification, and the t statistics accordingly much greater. Our conclusion is that it is possible to obtain relatively precise estimates of the long-run impact of income and price, even though multicollinearity prevents us from deriving precise short-run estimates.

A11.8 Show how this model may be derived from the demand function and the adaptive expectations process.

The adaptive expectations process may be rewritten:

\[ \hat{e}_{t+1} = \lambda \hat{e}_t + (1 - \lambda) \hat{e}_t. \]

Substituting this into (1), one obtains:

\[ B_t = \beta_1 + \beta_2 \lambda \hat{e}_t + \beta_2 (1 - \lambda) \hat{e}_t + u_t. \]

We note that if we lag (1) by one time period:

\[ B_{t-1} = \beta_1 + \beta_2 \hat{e}_t + u_{t-1}. \]

Hence:

\[ \beta_2 \hat{e}_t = B_{t-1} - \beta_1 - u_{t-1}. \]

Substituting this into the second equation above, one has:

\[ B_t = \beta_1 \lambda + \beta_2 \lambda \hat{e}_t + (1 - \lambda) B_{t-1} + u_t - (1 - \lambda) u_{t-1}. \]
11. Models using time series data

This is equation (3) in the question, with $\gamma_1 = \beta_1 \lambda$, $\gamma_2 = \beta_2 \lambda$, $\gamma_3 = 1 - \lambda$, and $v_t = u_t - (1 - \lambda)u_{t-1}$.

Explain why inconsistent estimates of the parameters will be obtained if equation (3) is fitted using ordinary least squares (OLS). (A mathematical proof is not required. Do not attempt to derive expressions for the bias.)

In equation (3), the regressor $B_{t-1}$ is partly determined by $u_{t-1}$. The disturbance term $v_t$ also has a component $u_{t-1}$. Hence the requirement that the regressors and the disturbance term be distributed independently of each other is violated. The violation will lead to inconsistent estimates because the regressor and the disturbance term are contemporaneously correlated.

Describe a method for fitting the model that would yield consistent estimates.

If the first equation in this exercise is true for time period $t + 1$, it is true for time period $t$:

$$i^e_t = \lambda i_{t-1} + (1 - \lambda) i^e_{t-1}.$$

Substituting into the second equation in (a), we now have:

$$B_t = \beta_1 + \beta_2 \lambda i_t + \beta_2 \lambda (1 - \lambda) i_{t-1} + (1 - \lambda) i^e_{t-1} + u_t.$$

Continuing to lag and substitute, we have:

$$B_t = \beta_1 + \beta_2 \lambda i_t + \beta_2 \lambda (1 - \lambda) i_{t-1} + \cdots + \beta_2 \lambda (1 - \lambda)^{s-1} i_{t-s+1} + (1 - \lambda)^s i^e_{t-s+1} + u_t.$$

For $s$ large enough, $(1 - \lambda)^s$ will be so small that we can drop the unobservable term $i^e_{t-s+1}$ with negligible omitted variable bias. The disturbance term is distributed independently of the regressors and hence we obtain consistent estimates of the parameters. The model should be fitted using a nonlinear estimation technique that takes account of the restrictions implicit in the specification.

Suppose that $u_t$ were subject to the first-order autoregressive process:

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where $\varepsilon_t$ is not subject to autocorrelation. How would this affect your answer to the second part of this question?

$v_t$ is now given by:

$$v_t = u_t - (1 - \lambda)u_{t-1} = \rho u_{t-1} + \varepsilon_t - (1 - \lambda)u_{t-1} = \varepsilon_t - (1 - \rho - \lambda)u_{t-1}.$$

Since $\rho$ and $\lambda$ may reasonably be assumed to lie between 0 and 1, it is possible that their sum is approximately equal to 1, in which case $v_t$ is approximately equal to the innovation $\varepsilon_t$. If this is the case, there would be no violation of the regression assumption described in the second part of this question and one could use OLS to fit (3) after all.

Suppose that the true relationship was actually:

$$B_t = \beta_1 + \beta_2 i_t + u_t \quad (1*)$$

with $u_t$ not subject to autocorrelation, and the model is fitted by regressing $B_t$ on $i_t$ and $B_{t-1}$, as in equation (3), using OLS. How would this affect the regression results?
The estimators of the coefficients will be inefficient in that $B_t - 1$ is a redundant variable. The inclusion of $B_t - 1$ will also give rise to finite sample bias that would disappear in large samples.

How plausible do you think an adaptive expectations process is for modelling expectations in a bond market?

The adaptive expectations model is implausible since the expectations process would change as soon as those traders taking advantage of their knowledge of it started earning profits.

A11.9 The regression indicates that the short-run income, price, and population elasticities for expenditure on food are $0.14$, $-0.10$, and $-0.05$, respectively, and that the speed of adjustment is $(1 - 0.73) = 0.27$. Dividing by $0.27$, the long-run elasticities are $0.52$, $-0.37$, and $-0.19$, respectively. The income and price elasticities seem plausible. The negative population elasticity makes no sense, but it is small and insignificant. The estimates of the short-run income and price elasticities are likewise not significant, but this is not surprising given that the point estimates are so small.

A11.10 The table gives the result of the specification with a lagged dependent variable for all the categories of expenditure.

<table>
<thead>
<tr>
<th>Category</th>
<th>LGDPI</th>
<th>LGP</th>
<th>LGPOP</th>
<th>LGCAT(-1)</th>
<th>Long-run effects</th>
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<td>ADM</td>
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<td>0.18</td>
<td>-0.10</td>
<td>0.06</td>
<td>2.03</td>
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<td>-0.21</td>
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<tr>
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<td>-0.40</td>
<td>0.07</td>
<td>0.01</td>
</tr>
<tr>
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<td>0.21</td>
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<td>0.07</td>
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<tr>
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<td>-0.10</td>
<td>0.06</td>
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<tr>
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<td>0.22</td>
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<td>0.14</td>
<td>0.00</td>
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<td>0.14</td>
<td>-0.12</td>
<td>0.05</td>
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<tr>
<td>TOYS</td>
<td>0.31</td>
<td>0.11</td>
<td>-0.27</td>
<td>0.08</td>
<td>1.44</td>
</tr>
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</table>
In his classic study Distributed Lags and Investment Analysis (1954), Koyck investigated the relationship between investment in railcars and the volume of freight carried on the US railroads using data for the period 1884–1939. Assuming that the desired stock of railcars in year $t$ depended on the volume of freight in year $t - 1$ and year $t - 2$ and a time trend, and assuming that investment in railcars was subject to a partial adjustment process, he fitted the following regression equation using OLS (standard errors and constant term not reported):

$$
\hat{I}_t = 0.077F_{t-1} + 0.017F_{t-2} - 0.0033t - 0.110K_{t-1} \\
R^2 = 0.85
$$

where $I_t = K_t - K_{t-1}$ is investment in railcars in year $t$ (thousands), $K_t$ is the stock of railcars at the end of year $t$ (thousands), and $F_t$ is the volume of freight handled in year $t$ (ton-miles).

Provide an interpretation of the equation and describe the dynamic process implied by it. (Note: It is best to substitute $K_t - K_{t-1}$ for $I_t$ in the regression and treat it as a dynamic relationship determining $K_t$).

Given the information in the question, the model may be written:

$$
K_t^* = \beta_1 + \beta_2F_{t-1} + \beta_3F_{t-2} + \beta_4t + u_t
$$

$$
K_t - K_{t-1} = I_t = \lambda(K_t^* - K_{t-1}).
$$

Hence:

$$
I_t = \lambda\beta_1 + \lambda\beta_2F_{t-1} + \lambda\beta_3F_{t-2} + \lambda\beta_4t - \lambda K_{t-1} + \lambda u_t.
$$

From the fitted equation:

$$
\hat{\lambda} = 0.110
$$

$$
\hat{\beta}_2 = \frac{0.077}{0.110} = 0.70
$$

$$
\hat{\beta}_3 = \frac{0.017}{0.110} = 0.15
$$

$$
\hat{\beta}_4 = \frac{-0.0033}{0.110} = -0.030.
$$

Hence the short-run effect of an increase of 1 million ton-miles of freight is to increase investment in railcars by 7,000 one year later and 1,500 two years later. It does not make much sense to talk of a short-run effect of a time trend.

In the long-run equilibrium, neglecting the effects of the disturbance term, $K_t$ and $K_t^*$ are both equal to the equilibrium value $\bar{K}$ and $F_{t-1}$ and $F_{t-2}$ are both equal to their equilibrium value $\bar{F}$. Hence, using the first equation:

$$
\bar{K} = \beta_1 + (\beta_2 + \beta_3)\bar{F} + \beta_4t.
$$

Thus an increase of one million ton-miles of freight will increase the stock of railcars by 940 and the time trend will be responsible for a secular decline of 33 railcars per year.
A11.12 One researcher asserts that consistent estimates will be obtained if (2) is fitted using OLS and (1) is fitted using IV, with \( Y_{t-1} \) as an instrument for \( X_t \). Determine whether this is true.

(2) may indeed be fitted using OLS. Strictly speaking, there may be an element of bias in finite samples because of noncontemporaneous correlation between \( v_t \) and future values of \( Y_t \).

We could indeed use \( Y_{t-1} \) as an instrument for \( X_t \) in (1) because \( Y_{t-1} \) is a determinant of \( X_t \) but is not (contemporaneously) correlated with \( u_t \).

The other researcher asserts that consistent estimates will be obtained if both (1) and (2) are fitted using OLS, and that the estimate of \( \beta_2 \) will be more efficient than that obtained using IV. Determine whether this is true.

This assertion is also correct. \( X_t \) is not correlated with \( u_t \), and OLS estimators are more efficient than IV estimators when both are consistent. Strictly speaking, there may be an element of bias in finite samples because of noncontemporaneous correlation between \( u_t \) and future values of \( X_t \).