Using Calculus to Model the Natural World

1. Human population growth between the years 1800 and 2000 can be described using a simple exponential model. The change in population, \( P \), over time, \( t \), is dependent on a constant \( r \):

\[
\frac{dP}{dt} = rP
\]

The equation tells us that not only was the population growing, but that the rate of growth increased until around the year 2000.

2. Predator-prey relationships in the natural world can also be modelled using calculus.

The Canadian lynx feeds almost exclusively on the snowshoe hare. The changing population of these two species shows a close-to-perfect cycle of alternating ups and downs. Conservationists can model this change using a pair of differential equations known as the Lotka-Volterra equations, or the predator-prey equations. If \( L \) is the number of lynx, and \( H \) is the number of hare, we get:

\[
\frac{dH}{dt} = aH - bHL
\]

\[
\frac{dL}{dt} = cHL - dL
\]

Look at these equations and think about how a change in one population will impact the other. Can you see how we get the repeating ups and downs in the graph shown above?
A Level Maths Formulae

Quadratic Equations

\[ ax^2 + bx + c = 0 \text{ has roots } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Laws of Logarithms

\[ \log_a x + \log_a y = \log_a (xy) \]
\[ \log_a x \log_b (xy) = \log_a (x^b) \]
\[ k \log_a x = \log_a (x^k) \]
\[ \log_a x = -\log_a (x^{-1}) \]
\[ x = a^\log_a x \text{ for } a > 0 \text{ and } x > 0 \]

Trigonometry

In the triangle $ABC$

- Sine rule: \[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]
- Cosine rule: \[ a^2 = b^2 + c^2 - 2bc \cos A \]
- Area: \[ \frac{1}{2} \sin \theta \]
- $\cos^2 A + \sin^2 A = 1$
- $\sec^2 A = 1 + \tan^2 A$
- $\cos 2A = 2 \cos^2 A - 1$
- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = 2 \cos^2 A - 1$
- $\sin \theta = \frac{b}{c}$
- $\cos \theta = \frac{a}{c}$

Calculus and Differential Equations

**Differentiation**

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\sin kx$</td>
<td>$k \cos kx$</td>
</tr>
<tr>
<td>$\cos kx$</td>
<td>$-k \sin kx$</td>
</tr>
<tr>
<td>$e^{kx}$</td>
<td>$ke^{kx}$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$f(x) + g(x)$</td>
<td>$f'(x) + g'(x)$</td>
</tr>
<tr>
<td>$f(x)g(x)$</td>
<td>$f'(x)g(x) + f(x)g'(x)$</td>
</tr>
<tr>
<td>$fg(x)$</td>
<td>$f'(x)g(x)'$</td>
</tr>
</tbody>
</table>

**Integration**

<table>
<thead>
<tr>
<th>Function</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1}$</td>
</tr>
<tr>
<td>$\sin kx$</td>
<td>$-\frac{\cos kx}{k} + c$</td>
</tr>
<tr>
<td>$\cos kx$</td>
<td>$\frac{\sin kx}{k} + c$</td>
</tr>
<tr>
<td>$e^{kx}$</td>
<td>$\frac{e^{kx}}{k} + c$</td>
</tr>
<tr>
<td>$\ln</td>
<td>x</td>
</tr>
<tr>
<td>$f(x) + g(x)$</td>
<td>$f(x) + g(x) + c$</td>
</tr>
<tr>
<td>$f(x)g(x)$</td>
<td>$f(x)g(x) + c$</td>
</tr>
<tr>
<td>$fg(x)$</td>
<td>$f(x)g(x) + c$</td>
</tr>
</tbody>
</table>

Mensuration

Area and Circumference of circle, radius $r$ and diameter $d$:

- $C = 2\pi r$
- $A = \pi r^2$

Pythagoras’ Theorem:

In any right-angled triangle where $a$, $b$, and $c$ are the length of the sides and $c$ is the hypotenuse:

\[ c^2 = a^2 + b^2 \]

Area of a trapezium:

\[ \text{Area} = \frac{1}{2} (a + b) h \]

where $a$ and $b$ are the lengths of the parallel sides and $h$ is their perpendicular separation.

Volume of a prism:

\[ \text{Area of cross section} \times \text{length} \]

For a circle of radius $r$, where an angle at the centre of $\theta$ radians subtends an arc of length $l$ and encloses an associated sector of area $a$:

\[ l = r\theta \quad a = \frac{1}{2} r^2 \theta \]

Sequences

General term of an arithmetic progression:

\[ u_n = a + (n-1)d \]

General term of a geometric progression:

\[ u_n = ar^{n-1} \]

Coordinate Geometry

A straight line graph, gradient $m$ passing through $(x_1, y_1)$ has equation

\[ y - y_1 = m (x - x_1) \]

Straight lines with gradients $m_1$ and $m_2$ are perpendicular when

\[ m_1 m_2 = -1 \]

Laws of Indices

\[ a^m \cdot a^n = a^{m+n} \]
\[ a^m \div a^n = a^{m-n} \]
\[ (a^m)^n = a^{mn} \]

Vectors

\[ \sqrt{(x + y)^2 + (z + k)^2} \]
\[ (x, y, z) \cdot (a, b, c) = xa + yb + zc \]