Introduction
The highest mountain in the world is Mount Everest, located in the Himalayas. Its peak is now measured to be 8848 metres above sea level.

The mountain was first climbed in 1953, by Edmund Hilary and Sherpa Tenzing, almost 100 years after its height was first measured as part of the Great Trigonometrical Survey of India in 1856. The original surveyors, who included George Everest, obtained a height of 8840 metres by measuring the distance and angle of elevation between Mount Everest and a fixed location.

What’s the point?
Once a right-angled triangle is seen in a particular problem then a mathematician only needs two pieces of information to be able to calculate all the other lengths and angles.

Objectives
By the end of this chapter you will have learned how to ...

- Use the formulae for Pythagoras’ theorem: \( a^2 + b^2 = c^2 \).
- Use the trigonometric ratios and apply them to find angles and lengths in right-angled triangles:
  \[
  \sin \theta = \frac{\text{Opp}}{\text{Hyp}}, \quad \cos \theta = \frac{\text{Adj}}{\text{Hyp}}, \quad \tan \theta = \frac{\text{Opp}}{\text{Adj}}
  \]
- Know the exact values of \( \sin \theta \) and \( \cos \theta \) for \( \theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ \) and \( 90^\circ \).
- Know the exact value of \( \tan \theta \) for \( \theta = 0^\circ, 30^\circ, 45^\circ \) and \( 60^\circ \).
- Write column vectors and draw vector diagrams.
- Add, subtract and find multiples of vectors.
Check in
1  Work out each of these.
   a  $7^2$  b  $4^2 + 6^2$  c  $3^2 + 5^2$
   d  $8^2 - 4^2$  e  $7^2 - 2^2$  f  $\sqrt{17^2 - 15^2}$
2  Rearrange these equations to make $x$ the subject.
   a  $y = \frac{x}{6}$  b  $y = \frac{x}{5}$  c  $y = \frac{x}{10}$
   d  $y = \frac{2}{x}$  e  $y = \frac{5}{x}$  f  $y = \frac{8}{x}$

Chapter investigation
An engineering company is building a ski lift.
The height of the lift is exactly 45 m.
An engineer suggests using a cable of length 200 m.
The maximum angle of incline is 12°.
Does the engineer’s lift meet the criteria?
Design a lift that meets the criteria using the smallest possible length of cable.
In a right-angled triangle the **hypotenuse** is the longest side. It is opposite the right angle.

- **Pythagoras’ theorem** states
  For any right-angled triangle, \( c^2 = a^2 + b^2 \)
  where \( c \) is the hypotenuse.

### EXAMPLE

**Label the sides.**
\[
\begin{align*}
& a = 5, \ b = 12, \ c = ? \\
& c^2 = a^2 + b^2 \\
& c^2 = 5^2 + 12^2 \\
& c^2 = 25 + 144 \\
& c^2 = 169 \\
& c = \sqrt{169} = 13 \text{ cm}
\end{align*}
\]

You can use Pythagoras’ theorem to find any side given two other sides.

- To find the hypotenuse use \( c^2 = a^2 + b^2 \)
- To find a shorter side use \( a^2 = c^2 - b^2 \) or \( b^2 = c^2 - a^2 \)

### EXAMPLE

**Calculate the unknown length in these right-angled triangles.**

**a**

- Label the sides.
  \[
  \begin{align*}
  & a = ?, \ b = 3, \ c = 5 \\
  & a^2 = c^2 - b^2 \\
  & a^2 = 5^2 - 3^2 \\
  & = 25 - 9 \\
  & a^2 = 16 \\
  & a = \sqrt{16} = 4 \text{ cm}
  \end{align*}
  \]

**b**

- Label the sides.
  \[
  \begin{align*}
  & a = 10, \ b = ?, \ c = 17 \\
  & b^2 = c^2 - a^2 \\
  & b^2 = 17^2 - 10^2 \\
  & = 289 - 100 \\
  & = 189 \\
  & b = \sqrt{189} = 13.7 \text{ cm (to 1 dp)}
  \end{align*}
  \]

If all three sides of a right-angled triangle are whole numbers, then the numbers make a Pythagorean triple.

Remember that \( c \) is the longest side.

Round the answer when it is not exact. One decimal place is sensible here.
Exercise 19.1S

1. Calculate the area of these squares. State the units of your answers.

   a  8 cm
   b  10 m
   c  1.8 m
   d  36 mm

2. Calculate the length of a side of these squares. State the units of your answers.

   a  Area = $81 \text{ m}^2$
   b  Area = $4 \text{ cm}^2$
   c  Area = $196 \text{ cm}^2$
   d  Area = $7.29 \text{ m}^2$

3. Calculate the unknown area for these right-angled triangles.

   a  Area = $12 \text{ cm}^2$
   b  Area = $8 \text{ cm}^2$
   c  Area = $10 \text{ cm}^2$
   d  Area = $20 \text{ cm}^2$

4. Find the length of the hypotenuse in each of these right-angled triangles.

   a  4 cm
   b  15 cm

5. In some of the triangles in question 4, all three sides have integer (whole number) values. Such sets of three numbers are called Pythagorean triples. List the Pythagorean triples from question 4.

6. Find the length of the missing side in each of these right-angled triangles.

   a  6 cm
   b  7 cm
   c  24 cm
   d  12 cm
   e  3.7 cm
   f  5.2 cm
   g  4.8 cm
   h  10 cm

7. Some of the triangles in question 6 are Pythagorean triples.

   a  Write down the Pythagorean triples in question 6.
   b  Compare these with your answers to question 5.
   c  Comment on anything you notice.
### 19.1 Pythagoras’ theorem

**Recap**
- **Pythagoras’ theorem** states that for any right-angled triangle, \( c^2 = a^2 + b^2 \) where \( c \) is the hypotenuse.
- To find the length of a shorter side use \( a^2 = c^2 - b^2 \) or \( b^2 = c^2 - a^2 \).

**To solve a problem using Pythagoras’ theorem**
1. Sketch a diagram. Label the right angle and the sides \( a, b \) and \( c \).
2. Substitute the values into the formula.
3. Round your answer to a suitable degree of accuracy and include any units.

**Example**
Calculate the length of the line segment from \((2, 4)\) to \((5, 2)\).

1. \( a = 2, b = 3, c = ? \)
2. \( c^2 = a^2 + b^2 \)
   \( c^2 = 4 + 9 \)
   \( c^2 = 13 \)
3. \( c = \sqrt{13} \)
   \( = 3.60555 \)
   \( = 3.6 \text{ units (to 1 dp)} \)

**Example**
A rectangle measures 5 cm by 8 cm. Find the length of its diagonal.

1. Mark all the facts you know on your diagram.
   - Diagonal, \( d \), is the hypotenuse of a right-angled triangle.
2. \( d^2 = 5^2 + 8^2 \)
   \( d^2 = 89 \)
3. \( d = 9.43 \text{ cm (2 dp)} \)

**Example**
A ladder of length 5.5 m leans against a wall. The foot of the ladder is 1 m from the wall. How far up the wall does the ladder reach?

1. The ladder makes a right-angled triangle with the wall.
   \( a = 1, b = ?, c = 5.5 \)
2. Use the formula \( b^2 = c^2 - a^2 \)
   \( b^2 = 5.5^2 - 1^2 \)
   \( b^2 = 29.25 \)
   \( b = 5.4 \text{ m} \)

**Geometry** Pythagoras and trigonometry
Exercise 19.1A

1. Calculate the distance between these points. Give your answers to a suitable degree of accuracy.
   a. (1, 2) and (4, 6)
   b. (2, 2) and (6, 5)
   c. (1, 2) and (2, 5)
   d. (0, 5) and (4, 1)
   e. (3, 6) and (6, 0)
   f. (−3, 7) and (3, 2)

2. Find the length of the diagonal of each rectangle.
   a. 2.4 cm
   b. 6 cm
   c. 5.2 cm

3. A rectangle has one side 4 cm and diagonal 10.4 cm. Find the length of the other side.

4. Find the length of the diagonal of a square with side length 8 cm.

5. Find the length of the side of a square with diagonal length 8 cm.

6. An isosceles triangle has base 10 cm. The other two sides of the triangle are each 13 cm.
   a. Find the height of the triangle.
   b. Find the area of the triangle.

7. The top of a 4-metre ladder leans against the top of a wall. The wall is 3.8 metres high. How far from the wall is the bottom of the ladder?

8. The diagram shows a path across a rectangular field. How much further is it from A to C along the sides of the field than along the path?

9. PQR andPRS are right-angled triangles. Find the length PS.

10. ABC and ACD are right-angled triangles. Find the length AB.

11. A square and an isosceles triangle are joined together. Find the total perimeter of the new shape.

12. Prove that a triangle with sides 20 cm, 21 cm and 29 cm is right-angled.

13. Jeremy draws this triangle.
   a. Explain why angle x cannot be 90°.
   b. Is angle x acute or obtuse? Use a sketch to explain your answer.

14. A Pythagorean triple is three whole numbers (a, b, c) that satisfy \( a^2 + b^2 = c^2 \).
   (3, 4, 5) and (6, 8, 10) are Pythagorean triples. (3, 4, 5) is a primitive Pythagorean triple. (6, 8, 10) is not a primitive Pythagorean triple, as it is a multiple of (3, 4, 5).
   There are seven primitive Pythagorean triples with \( c < 50 \). Can you find them all?
Trigonometry 1

In a right-angled triangle the longest side is the **hypotenuse**. The side next to the labelled angle is called the **adjacent**. The side opposite the labelled angle is called the **opposite**.

For a right-angled triangle with angle \( \theta \), the ratio \( \frac{\text{opposite side}}{\text{adjacent side}} \) is constant. This ratio is called the **tangent ratio** or **tan**.

\[
\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}
\]

**EXAMPLE**

Calculate the value of \( \tan \theta \) in these triangles.

(a) \( \tan 21^\circ \)

\[
\tan 21^\circ = \frac{5}{13} = 0.38 \ (2 \text{ dp})
\]

(b) \( \tan 53^\circ \)

\[
\tan 53^\circ = \frac{2}{1.5} = 1.33 \ (2 \text{ dp})
\]

You can use the tan button on your calculator to work out the value of tan for any angle.

**EXAMPLE**

Work out the value of tan for each angle. Give your answers to 2 dp.

(a) \( \tan 21^\circ \)

\[
\tan 21^\circ = 0.383\ldots = 0.38 \ (2 \text{ dp})
\]

(b) \( \tan 53^\circ \)

\[
\tan 53^\circ = 1.327\ldots = 1.33 \ (2 \text{ dp})
\]

This means that you have these triangles.

You can use the inverse function \( \tan^{-1} \) to find the angle if you know the opposite and adjacent sides.

\[
\tan^{-1} \left( \frac{5}{13} \right) = 21.037\ldots = 21^\circ \ (2 \text{ sf})
\]

\[
\tan^{-1} \left( \frac{2}{1.5} \right) = 53.130\ldots = 53^\circ \ (2 \text{ sf})
\]

The ancient Egyptians used an early form of trigonometry for building pyramids.

Check that you can use the buttons on your calculator to work out the values of tan.
Exercise 19.2S

1. Draw a copy of each triangle. Label the sides of each triangle with hypotenuse, adjacent, opposite.

2. Use the tan button on your calculator to work out the value of tan for each angle. Give your answers to 2 dp.
   a. tan 28°  
   b. tan 32°  
   c. tan 23°  
   d. tan 67°  
   e. tan 37°  
   f. tan 56°

3. a. Work out the value of tan θ in each triangle. Give your answers to 2 dp.
   i.  
   ii.  
   iii.  
   iv.  
   v.  
   vi.  

b. Compare your answers to the values of tan in question 2. Write down the value of the unknown angle in each triangle.

c. Use the tan⁻¹ button on your calculator to check your answers to part b.

4. Use the tan⁻¹ button on your calculator to work out the size of the unknown angle in each triangle. Give your answer to 1 decimal place.

5. Check your answers to question 4 by constructing an accurate drawing of each triangle. Could you check your answers using an accurate scale drawing instead?
19.2 Trigonometry 1

**RECAP**
- In a right-angled triangle \( \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} \)

You can use the tan ratio to find the length of an unknown side in a right-angled triangle.

- To find the length of the opposite side, multiply the adjacent by \( \tan \theta \).
- To find the length of the adjacent, divide the opposite by \( \tan \theta \).

**HOW TO**
1. Draw a copy of the triangle. Label the sides: adjacent, hypotenuse and opposite.
2. Substitute the values that you know into the formula
   \[ \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} \]
3. Find the missing angle or length.

**EXAMPLE**
Find the missing sides.

1. [Diagram of a right-angled triangle with angles and sides labeled]
2. \( \tan 36^\circ = \frac{x}{15} \)
3. \( 15 \times \tan 36^\circ = x \)
   \( x \approx 10.9 \text{ cm (1 dp)} \)

**EXAMPLE**
Jan looks at a boat from the top of a 40-metre lighthouse. She says the boat is over 100 metres from the lighthouse. Is Jan correct?

State any assumptions you make.

1. Draw a sketch to show more information.
2. \( \tan 20^\circ = \frac{40}{d} \)
   Using \( \tan \theta = \frac{O}{A} \)
   \( d = \frac{40}{\tan 20^\circ} \)
   Rearrange to find \( d \).
3. \( d = 109.89... \text{ m} \)
   This is over 100 m, so Jan is correct.

Some values of tan have exact answers:
- \( \tan 30 = \frac{1}{\sqrt{3}} \)
- \( \tan 45 = 1 \)
- \( \tan 60 = \sqrt{3} \)

You need to learn these for your exam.

Use a letter like \( d \) for the distance you want to find.

Alternate angle from parallel horizontal lines.
Exercise 19.2A

1. Find the missing side in each triangle. Give your answers to 3 significant figures.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

2. Find the missing side in each triangle. Give your answers to 3 significant figures.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

3. A boat is observed from the top of a 50-metre lighthouse. How far is the boat from the cliff?

4. Carl looks at Blackpool Tower from the beach. He estimates these measurements.
   a. Use Carl’s estimates to find the height of Blackpool Tower.
   b. Why might the answer to a be inaccurate?

5. Beth is 1.6 m tall. She looks at the top of a statue that is 8 metres from her. Beth says the statue is over 10 metres tall. Is she correct? Explain your answer.

6. Here is an isosceles right-angled triangle.
   a. Explain why the angle in the triangle is 45°.
   b. Use the triangle to explain why \( \tan 45° = 1 \).

7. Here is an equilateral triangle with sides 2 cm.
   The triangle is folded in half.
   a. Find \( h \). Give your answer as an exact square root.
   b. Use the triangle to explain why \( \tan 60° = \sqrt{3} \).
   c. Use the triangle to explain why \( \tan 30° = \frac{1}{\sqrt{3}} \).

*8. The diagram shows a flag on top of a building. How tall is the flag?
The **tangent** ratio is: \( \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} \)

There are two other ratios you can use in **right-angled triangles**. The **sine** ratio (\( \sin \)) connects the opposite side and the hypotenuse. The **cosine** ratio (\( \cos \)) connects the adjacent side and the hypotenuse.

- **Sine ratio**
  \[ \sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} \]

- **Cosine ratio**
  \[ \cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \]

A mnemonic that helps you remember the trigonometry ratios is:

Sir Oliver’s Horse, Came Ambling Home, To Oliver’s Aunt

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}
\]

You can use the inverse functions \( \sin^{-1} \), \( \cos^{-1} \) and \( \tan^{-1} \) to find the angle if you know two sides.

**EXAMPLE**

Find the missing angles. Give your answers to the nearest degree.

- **a**
  \[
  \text{You have adjacent and hypotenuse, so use cosine.}
  \]
  \[
  \cos x = \frac{6}{10.5}
  \]
  \[
  x = \cos^{-1} \left( \frac{6}{10.5} \right)
  \]
  \[
  x = 55^\circ \quad \text{(to the nearest degree)}
  \]

- **b**
  \[
  \text{You have opposite and hypotenuse, so use sine.}
  \]
  \[
  \sin y = \frac{7.5}{11.7}
  \]
  \[
  y = \sin^{-1} \left( \frac{7.5}{11.7} \right)
  \]
  \[
  y = 40^\circ \quad \text{(to the nearest degree)}
  \]

- **c**
  \[
  \text{You have opposite and adjacent, so use tan.}
  \]
  \[
  \tan z = \frac{5.2}{11.1}
  \]
  \[
  z = \tan^{-1} \left( \frac{5.2}{11.1} \right)
  \]
  \[
  z = 25^\circ \quad \text{(to the nearest degree)}
  \]
Exercise 19.3S

1. Use the cos button on your calculator to work out the value of cos for each angle. Give your answers to 2 dp.
   a. cos 20°
   b. cos 35°
   c. cos 50°
   d. cos 72°
   e. cos 100°
   f. cos 140°

2. Use the cos⁻¹ button on your calculator to work out each calculation. Give your answer to the nearest whole number.
   a. cos⁻¹(0.94)
   b. cos⁻¹(0.82)
   c. cos⁻¹(0.64)
   d. cos⁻¹(0.31)
   e. cos⁻¹(−0.17)
   f. cos⁻¹(−0.77)

3. Use the sin button on your calculator to work out the value of sin for each angle. Give your answers to 2 dp.
   a. sin 20°
   b. sin 35°
   c. sin 50°
   d. sin 72°
   e. sin 100°
   f. sin 140°

4. Use the sin⁻¹ button on your calculator to work out each calculation. Give your answer to the nearest whole number.
   a. sin⁻¹(0.34)
   b. sin⁻¹(0.57)
   c. sin⁻¹(0.77)
   d. sin⁻¹(0.95)
   e. sin⁻¹(0.98)
   f. sin⁻¹(0.64)

5. You can write the values of sin and cos for 90°, 60°, 45°, 30° and 0° exactly. Match each calculation with one of the exact answers on the cards.
   
   ![Card Options]

   a. cos 90°
   b. sin 90°
   c. cos 60°
   d. sin 60°
   e. cos 45°
   f. sin 45°
   g. cos 30°
   h. sin 30°
   i. cos 0°
   j. sin 0°

6. Label the hypotenuse and the adjacent side in each triangle. Use cos to find the missing angle in each triangle.

   ![Triangle Images]

7. Label the hypotenuse and the opposite side in each triangle. Use sin to find the missing angle in each triangle.

   ![Triangle Images]

8. Label the adjacent side, opposite side and hypotenuse in each triangle. Find the missing angle. Give your answer to 1 dp.

   ![Triangle Images]
19.3 Trigonometry 2

RECAP
- The tan ratio connects the opposite side and the adjacent side.
- The sin ratio connects the opposite side and the hypotenuse.
- The cos ratio connects the adjacent side and the hypotenuse.

HOW TO
1. Label the sides adjacent, hypotenuse and opposite.
   Decide which ratio to use.
2. Substitute the values that you know into one of the formulae
   \[
   \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}
   \]
3. Find the missing angle or length.

Find the missing sides.

A \( \triangle \) with angle \( 22^\circ \) and side \( b \) is given.

- **Diagram and Calculation:**
  - **Step 1:** Use the cosine ratio to find \( b \).
  - **Step 2:** Use the sine ratio to find \( h \).

**Example:**

**ABC\( \text{D}\) is a parallelogram. \(AB = 8.2\text{ cm}, BC = 6.6\text{ cm}\) and angle \(ABC = 53^\circ\). Work out the area of the parallelogram.

- **Diagram and Calculation:**
  - **Step 1:** Sketch a diagram.
  - **Step 2:** Find the perpendicular height, \( x \).
  - **Step 3:** Calculate the area.

Some values of cos and sin have exact answers.

- \( \cos 45^\circ = \frac{\sqrt{2}}{2} \) and \( \sin 45^\circ = \frac{\sqrt{2}}{2} \)
- \( \cos 30^\circ = \frac{\sqrt{3}}{2} \) and \( \sin 30^\circ = \frac{1}{2} \)
- \( \cos 60^\circ = \frac{1}{2} \) and \( \sin 60^\circ = \frac{\sqrt{3}}{2} \)

You need to learn these for your exam.
Exercise 19.3A

1. Find the missing side in each of these right-angled triangles. Give your answers to 3 significant figures.

   a. 
   b. 
   c. 
   d. 
   e. 
   f. 
   g. 
   h. 
   i. 

2. A parallelogram has sides of length 5 cm and 9 cm. The smaller angles are both 45°. Find the area of the parallelogram.

3. A rhombus has side length 8 cm and smaller angle 35°. Find the area of the rhombus.

4. Here is an isosceles right-angled triangle.

   a. Explain why the length of the hypotenuse is \( \sqrt{2} \) cm.
   b. Use the triangle to explain why \( \cos 45 = \frac{1}{\sqrt{2}} \) and \( \sin 45 = \frac{1}{\sqrt{2}} \).

5. Here is an equilateral triangle with sides 2 cm. The triangle is folded in half.

   a. Use the triangle to explain why \( \cos 60 = \frac{1}{2} \) and \( \sin 60 = \frac{\sqrt{3}}{2} \).
   b. Use the triangle to explain why \( \cos 30 = \frac{\sqrt{3}}{2} \) and \( \sin 30 = \frac{1}{2} \).

6. A see-saw is 2.8 m long. When one end touches the ground it makes an angle of 35° with it.

   Tina says the other end is less than \( 1\frac{1}{2} \) metres above the ground. Is she correct?

7. After take-off, a plane climbs at an angle of 32° to the horizontal. Find the height of the plane above the ground when it has travelled 1 kilometre.

8. The angle between the legs of a stepladder must be 74°. A metal bar is needed to keep the legs the correct distance apart. Is a 2 metre bar long enough?

   \[ \text{bar} \]
SKILLS

19.4 Vectors

You describe a translation by a **vector**. You specify the distance moved left or right, and then up or down.

The triangle is translated by the vector \( \begin{pmatrix} 4 \\ 3 \end{pmatrix} \).

- A vector has a length and a direction.

You can draw a vector as an arrowed line. Its orientation gives the direction of movement; its length gives the distance.

These lines are all parallel and the same length. They all represent the same vector \( \mathbf{a} \).

You can tie a vector to a starting point. The vector \( -\mathbf{a} \) is parallel, the same length, in the opposite direction to \( \mathbf{a} \).

The line \( ST \) represents the vector \( \overrightarrow{ST} = \mathbf{s} \).

- You can multiply a vector by a number.

The vector \( 2\mathbf{p} \) is parallel to the vector \( \mathbf{p} \) and twice the length.

The vector \( 3\mathbf{p} \) is parallel to the vector \( \mathbf{p} \) and three times the length.

The vector \( -\mathbf{p} \) is parallel to the vector \( \mathbf{p} \) and the same length, but in the opposite direction.

- Vectors represented by parallel lines are multiples of each other.

- Vectors can be described using the notation \( \overrightarrow{AB} \) or bold type, \( \mathbf{a} \).

- In handwriting, vectors can be shown with an underline, \( \underline{a} \).

You can add and subtract vectors by putting them 'nose to tail'. The result of the addition or subtraction is called the **resultant** vector.

\[
\mathbf{m} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\]

Calculate \( \mathbf{m} + \mathbf{n} \) and \( \mathbf{m} - \mathbf{n} \).

**Example**

\[
\begin{align*}
\mathbf{m} + \mathbf{n} & = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \\
\mathbf{m} - \mathbf{n} & = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}
\end{align*}
\]

- 5 right, 1 up
- 1 right, 3 up
Exercise 19.4S

1 Draw these vectors on squared paper.
   \[ \begin{align*}
   \mathbf{a} &= \binom{4}{3}, & \mathbf{b} &= \binom{2}{5}, & \mathbf{c} &= \binom{-1}{4}, \\
   \mathbf{d} &= \binom{-3}{-3}, & \mathbf{e} &= \binom{0}{2}, & \mathbf{f} &= \binom{-4}{0}, \\
   \mathbf{g} &= \binom{-1}{4}, & \mathbf{h} &= \binom{2}{5}, & \mathbf{i} &= \binom{-3}{-3}.
   \end{align*} \]

2 Use \( \overrightarrow{AB} \) notation to identify equal vectors in this diagram.

3 \( ABCDEF \) is a regular hexagon.
   \( X \) is the centre of the hexagon.
   \( \overrightarrow{XA} = \mathbf{a} \) and \( \overrightarrow{AB} = \mathbf{b} \).

4 \( JKLMNOPQ \) is a regular octagon.
   \( \overrightarrow{OJ} = \mathbf{j}, \overrightarrow{OM} = \mathbf{m}, \overrightarrow{OP} = \mathbf{p}. \)

5 If \( \mathbf{p} = \binom{3}{4} \) and \( \mathbf{q} = \binom{-2}{-5} \) draw these vectors on square grid paper.
   \[ \begin{align*}
   \mathbf{a} &= \mathbf{p} + \mathbf{q}, & \mathbf{b} &= \mathbf{p} + \mathbf{p}, \\
   \mathbf{c} &= \mathbf{p} - \mathbf{q}, & \mathbf{d} &= \mathbf{p} + \mathbf{p} + \mathbf{q}.
   \end{align*} \]

6 The diagram shows vectors \( \mathbf{s} \) and \( \mathbf{t} \).
   On square grid paper draw the vectors that represent
   \[ \begin{align*}
   \mathbf{a} &= \mathbf{s} + \mathbf{s}, & \mathbf{b} &= \mathbf{s} + \mathbf{t}, \\
   \mathbf{c} &= \mathbf{t} + \mathbf{s}, & \mathbf{d} &= \mathbf{s} - \mathbf{t}, \\
   \mathbf{e} &= \mathbf{t} - \mathbf{s}, & \mathbf{f} &= \mathbf{t} + \mathbf{t} - \mathbf{s}.
   \end{align*} \]

7 The diagram shows vectors \( \mathbf{g} \) and \( \mathbf{h} \).
   On isometric paper draw the vectors that represent
   \[ \begin{align*}
   \mathbf{a} &= \mathbf{g} + \mathbf{g}, & \mathbf{b} &= \mathbf{g} + \mathbf{h}, \\
   \mathbf{c} &= \mathbf{h} - \mathbf{g}, & \mathbf{d} &= \mathbf{g} - \mathbf{h} - \mathbf{h}.
   \end{align*} \]

8 Write down a vector that is
   \[ \begin{align*}
   \mathbf{a} &= \text{in the same direction as } \binom{-2}{1} \text{ but 4 times as long}, \\
   \mathbf{b} &= \text{half as long as } \binom{-6}{8} \text{ and in the opposite direction}.
   \end{align*} \]

9 \[ \begin{align*}
   \mathbf{a} &= \binom{-2}{4}, & \mathbf{b} &= \binom{3}{-1}, & \mathbf{c} &= \binom{0}{-2}.
   \end{align*} \]
   \[ \begin{align*}
   \mathbf{a} &= \text{Calculate } \begin{align*}
   i) \ a + b + c, & ii) \ 2a + b, \\
   iii) \ a - b - c, & iv) \ b - 3c, \\
   v) \ 2a + 3b, & vi) \ 3b + \frac{1}{2}a - \frac{1}{2}c.
   \end{align*} \]
   \[ \begin{align*}
   \mathbf{b} &= \text{Use diagrams to check your answers.}
   \end{align*} \]
### 19.4 Vectors

#### RECAP
- A vector has a fixed length and direction.
- You can add and subtract vectors by putting them ‘nose to tail’.
- You can extend addition to more than two vectors.

\[ \mathbf{b} + \mathbf{b} + \mathbf{b} + \mathbf{b} = 4\mathbf{b} \]

You can write this vector as 4\( \mathbf{b} \).
- Vectors represented by parallel lines are multiples of each other.

#### HOW TO
1. Draw a diagram and mark the known vectors.
2. Use your knowledge of vector addition and subtraction. Look out for parallel vectors.
3. Answer the question. Make sure you use the correct notation for vectors.

#### EXAMPLE

**OABC** is a parallelogram.

\[ \overrightarrow{OA} = \mathbf{a} \quad \overrightarrow{OC} = \mathbf{c} \]

Write the vector that represents the diagonal

\[ \mathbf{a} \quad \overrightarrow{OB} \quad \mathbf{b} \quad \overrightarrow{CA} \]

**EXAMPLE**

**OABC** is a trapezium.

The parallel sides \( \mathbf{CB} \) and \( \mathbf{OA} \) are such that

\( \mathbf{CB} = 3\mathbf{OA} \).

\( \overrightarrow{OA} = \mathbf{a} \quad \overrightarrow{OC} = \mathbf{b} \)

Write the vector \( \overrightarrow{AB} \) in terms of \( \mathbf{a} \) and \( \mathbf{b} \).

[Diagram of parallelogram and trapezium]
Exercise 19.4A

1 \( \text{OPQR} \) is a rectangle.  
\[ \overrightarrow{OP} = \mathbf{p} \text{ and } \overrightarrow{OR} = \mathbf{r}. \]
Work out the vector, in terms of \( \mathbf{p} \) and \( \mathbf{r} \), that represents
a \( \overrightarrow{PQ} \)  
b \( \overrightarrow{OQ} \)  
c \( \overrightarrow{QR} \)  
d \( \overrightarrow{RP} \)

2 \( \text{OJKL} \) is a rhombus.  
\[ \overrightarrow{OJ} = \mathbf{j} \text{ and } \overrightarrow{OL} = \mathbf{l}. \]
Work out the vector, in terms of \( \mathbf{j} \) and \( \mathbf{l} \), that represents
a \( \overrightarrow{JK} \)  
b \( \overrightarrow{JL} \)  
c \( \overrightarrow{OK} \)  
d \( \overrightarrow{KL} \)

3 \( \overrightarrow{OP} = \mathbf{p} \) and \( \overrightarrow{OQ} = \mathbf{q} \).

4 The diagram shows vectors \( \mathbf{x} \) and \( \mathbf{y} \).  
On square grid paper draw the vectors
a \( 2\mathbf{x} \)  
b \( 3\mathbf{y} \)  
c \( 2\mathbf{x} + 3\mathbf{y} \)  
d \( 3\mathbf{x} - \mathbf{y} \)  
e \( \mathbf{y} - 2\mathbf{x} \)  
f \( \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} \)  
g \( 2(\mathbf{x} + \mathbf{y}) \)  
h \( \frac{1}{2}(2\mathbf{x} - 3\mathbf{y}) \)  
i \( 3\mathbf{x} + 4\mathbf{y} \)

5 \( \text{OJKL} \) is a trapezium.  
The parallel sides \( \overrightarrow{OJ} \) and \( \overrightarrow{LK} \) are such that \( \overrightarrow{LK} = \frac{1}{4} \overrightarrow{OJ} \).  
\( \overrightarrow{OJ} = 6\mathbf{j} \)
Work out, in terms of \( \mathbf{j} \), the vectors that represent
a \( \overrightarrow{KL} \)  
b \( \overrightarrow{JK} \)  
c \( \overrightarrow{OK} \)  
d \( \overrightarrow{KL} \)

6 \( \text{ABCDEFGH} \) is a regular octagon.  
\( \overrightarrow{AB} = \mathbf{a} \) and \( \overrightarrow{DE} = \mathbf{d} \).
Write, and simplify, the vectors
a \( \overrightarrow{OG} \)  
b \( \overrightarrow{OL} \)  
c \( \overrightarrow{OK} \)  
d \( \overrightarrow{OB} \)  
e \( \overrightarrow{OA} \)  
f \( \overrightarrow{OC} \)  
g \( \overrightarrow{OB} \)  
h \( \overrightarrow{OF} \)  
i \( \overrightarrow{PE} \)  
j \( \overrightarrow{PD} \)  
k \( \overrightarrow{EF} \)  
l \( \overrightarrow{CA} \)  
m \( \overrightarrow{JK} \)  
n \( \overrightarrow{JL} \)  
o \( \overrightarrow{JE} \)  
p \( \overrightarrow{FD} \)  
q \( \overrightarrow{DF} \)  
r \( \overrightarrow{DC} \)  
s \( \overrightarrow{ME} \)  
t \( \overrightarrow{HG} \)  
u \( \overrightarrow{KD} \)

7 Show that
a \( \mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p} \)  
b \( 3(\mathbf{p} + \mathbf{q}) = 3\mathbf{p} + 3\mathbf{q} \)  
c \( -(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q} \)  
d \( (\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r}) \)
Summary

Checkout

You should now be able to...

Test it

Questions

✔️ Use the formulae for Pythagoras’ theorem: \( a^2 + b^2 = c^2 \).

✔️ Use the trigonometric ratios and apply them to find angles and lengths in right-angled triangles:

\[
\sin \theta = \frac{\text{Opp}}{\text{Hyp}} \quad \cos \theta = \frac{\text{Adj}}{\text{Hyp}} \quad \tan \theta = \frac{\text{Opp}}{\text{Adj}}
\]

✔️ Know the exact values of \( \sin \theta \) and \( \cos \theta \) for \( \theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ \) and tan \( \theta \) for \( \theta = 0^\circ, 30^\circ, 45^\circ \) and 60°.

✔️ Write column vectors and draw vector diagrams.

✔️ Add, subtract and find multiples of vectors.

Language

Meaning

Example

Hypotenuse
The side opposite the right angle in a right-angled triangle.

Pythagoras’ theorem
For a right-angled triangle, \( c^2 = a^2 + b^2 \) where \( c \) is the hypotenuse.

Adjacent
The side next to the labelled angle in a right-angled triangle.

Opposite
The side opposite the labelled angle in a right-angled triangle.

Sine ratio
The ratio of the length of the opposite side to the hypotenuse in a right-angled triangle.

Cosine ratio
The ratio of the length of the adjacent side to the hypotenuse in a right-angled triangle.

Tangent ratio
The ratio of the length of the opposite side to the adjacent side in a right-angled triangle.

Vector
A vector is a quantity with both size and direction.

Resultant
The vector that is equivalent to adding or subtracting two or more vectors.

Multiple
The original vector multiplied by an integer (a whole number).
# Review

1. Calculate the lengths $a$ and $b$ in these right-angled triangles. Give your answers to 1 decimal place.

   ![Diagram](image1)

2. These two triangles are similar. Calculate the lengths $x$ and $y$.

   ![Diagram](image2)

3. Write as a fraction the value of
   
   i. $\sin x$
   
   ii. $\cos x$
   
   iii. $\tan x$.

4. Calculate the lengths $a$, $b$ and $c$ to 3 significant figures.

   ![Diagram](image3)

5. Without using a calculator, select the correct value of each ratio.

   a. $\sin 60^\circ$
   
   b. $\cos 0^\circ$

6. A ship starts at a port then sails 30 km due south then 50 km due east.

   a. What is the shortest distance from the final position of the ship back to the port?
   
   b. What is the bearing of the ship from the port?

7. Work out these vector sums.

   a. $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$
   
   b. $\begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
   
   c. $5 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
   
   d. $2 \begin{pmatrix} 3 \\ -5 \end{pmatrix} + 5 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

8. Write down the column vectors to describe vectors $\mathbf{u}$ and $\mathbf{v}$.

---

## What next?

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Assessment 19

1 a Hannah says that the hypotenuse of this triangle is 181 cm.
   Explain Hannah’s error and work out the correct hypotenuse.
   [3 marks]

   b Paul says that the missing side in this triangle is 15.45 m to 2 dp.
   Explain Paul’s error and work out the correct value of the missing side.
   [4 marks]

2 a Angelina says that 2, 3, 13 is a Pythagorean triple.
   Explain why she is wrong.
   [2 marks]

   b Darshna says that 20, 99 and 100 is a Pythagorean triple.
   Explain which number is wrong and find the correct third value in the triple.
   [2 marks]

3 Laura plots the points, (−3, −5) and (6, −1).
   Find the distance between the points.
   [3 marks]

4 There are three rectangles with area 28 m².
   The sides of the rectangle are a whole number of metres.
   a Find the side lengths of the three possible rectangles.
      [3 marks]
   b Calculate the length of the diagonal of each of these rectangles.
      Give your answer to 4 sf.
      [6 marks]

5 Douglas draws five triangles with the following sides:
   a 9, 12, 15  b 9, 14, 17  c 1.6, 3.0, 3.4  d 11, 19, 22  e 3.6, 7.7, 8.5
   Which of these are right angled triangles? Explain your answers.
   [15 marks]

6 Ainslie is competing in a yacht race.
   He starts the harbour, H, and sails 2.5 km on a bearing of 062° until he reaches a buoy, B.
   He then sails for 3.6 km on a bearing of 152° until he reaches the lighthouse, L.
   He then returns to the Harbour.
   The race course is shown on the diagram.
   a Prove that the angle x is a right angle.
      [10 marks]
   b Calculate the distance from the lighthouse to the harbour.
   c Hence calculate the total length of the race.
   d The angle y is 55°. Hence find the bearing from L to H.

Geometry  Pythagoras and trigonometry
7 i Erica is given the three triangles shown. All lengths are in cm and angles in degrees correct to 1 decimal place.

She says incorrectly that to find the

- a angle in triangle a, you use tan
- b angle in triangle b, you use sin
- c angle in triangle c, you use cos.

Write correctly which trigonometric function you need to use to find each angle, and hence find the angles. [3 marks]

ii Use trigonometry to find the missing side in each triangle. [3 marks]

8 a The base of a ladder is on horizontal ground and is leaning against a vertical wall. The base is 1.75 m from the wall and the ladder makes an angle of 12.5° with the wall. How long is the ladder?

b How far up the wall does the ladder reach? [5 marks]

9 a Draw these vectors on squared paper:

i \[ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \]

ii \[ \begin{pmatrix} -5 \\ 12 \end{pmatrix} \]

iii \[ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \]

iv \[ \begin{pmatrix} -2 \\ -3 \end{pmatrix} \]

b If \( \begin{pmatrix} 4 \\ 6 \end{pmatrix} \) and \( \begin{pmatrix} -2 \\ 7 \end{pmatrix} \). Draw these vectors on squared paper:

i \( x + y \)

ii \( y - x \)

iii \( x - y \)

iv \( 2x \)

v \( -2y \)

vi \( 2x - 3y \)

c Use Pythagoras theorem to find the lengths of each of the vectors in part b. Leave your answers in exact form.

d Compare your drawings to parts ii and iii. Write down two things you notice. [46 marks]

10 Greg has a square grid. In his grid a vector one square to the right is given by \( x \) and one square vertically upwards by \( y \).

He writes down the vectors shown in terms of the vectors \( x \) and \( y \) as follows:

a \( 3x \)

b \( 2x + 7y \)

c \( 6y \)

d \( x - 3y \)

e \( 3x + 2y \)

f \( 3x + 7y \)

Which vectors has he written down correctly?

For the vectors he has written incorrectly, write down the correct vector in terms of the vectors \( x \) and \( y \). [10 marks]

11 If \( \begin{pmatrix} 4 \\ 1 \end{pmatrix} \), \( \begin{pmatrix} 7 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} -8 \\ 5 \end{pmatrix} \), write down these vectors:

a \( 5y \)

b \( -2z \)

c \( y + z \)

d \( y - x \)

e \( x + y + z \)

f \( 4z - 2x \)

g \( 4y + 3x - 2z \) [15 marks]
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