Bridging material

This chapter will remind you how to
- manipulate algebraic expressions and evaluate formulae
- solve linear equations and graph linear equations
- find the midpoint and length of a line segment
- solve simultaneous equations and deal with inequalities
- use circle theorems in problems
- use Pythagoras’ theorem and the trigonometric ratios.
### 0.1 Expanding and factorising

To simplify an algebraic expression, you collect like terms. Like terms have the same base and the same index.

2\(x^2\) and \(x^2\) are like terms. They both have base \(x\) and index 2.

2\(x^2\) and \(x^3\) are unlike terms. They contain different base letters.

#### Example 1

**Simplify**

- **a** \(3a - 2b + 4a - 3b + c\)
- **b** \(x - x^2 + 2x + 3x^2\)

- **a** Collect the like terms: 
  \(3a - 2b + 4a - 3b + c = 7a - 5b + c\)
- **b** Collect the like terms: 
  \(x - x^2 + 2x + 3x^2 = 3x + 2x^2\)

To expand double brackets, you multiply each term in the first bracket by each term in the second bracket.

\((x + 3)(x + 2) = x^2 + 2x + x + 2\)
\(= x^2 + 3x + 2\)

#### Example 2

**Expand and simplify**

- **a** \((x + 4)(x + 5)\)
- **b** \((2x - 1)(x - 3)\)

- **a** \((x + 4)(x + 5) = x^2 + 5x + 4x + 20\)
  \(= x^2 + 9x + 20\)
- **b** \((2x - 1)(x - 3) = 2x^2 - 6x - x + 3\)
  \(= 2x^2 - 7x + 3\)

The reverse of expanding is called factorising.

**Example 3**

You can sometimes factorise a quadratic expression into double brackets.

- **Factorise fully**
  - **a** \(2x + 8\)
  - **b** \(y^2 - 3y\)
  - **c** \(2x^2 + 4x\)

- **a** \(2x + 8\)
  \(= 2(x + 4)\)
- **b** \(y^2 - 3y\)
  \(= y(y - 3)\)
- **c** \(2x^2 + 4x\)
  \(= 2x(x + 2)\)

You may find it helpful to multiply the terms in this order:
- **F** Firsts \(\times \times \)
- **O** Outsers \(\times \times \)
- **I** Inners \(\times \times \)
- **L** Lasts \(\times \times \)

Remember the rules for multiplying negative terms.

- \(\times \times = \times \)
- \(\times \times = \times \)
- \(\times \times = \times \)
- \(\times \times = \times \)

The two numbers in the brackets multiply to give the constant and add to give the **coefficient** of \(x\).

\(x^2 + 2x - 8\)

- **Factorise fully** \(x^2 + 2x - 8\)

\(-1, 8; 1, -8; -2, 4; 2, -4\)
\(x^2 + 2x - 8 = (x - 2)(x + 4)\)

Check your answer by expanding the brackets.

### Example 4

- **Factorise fully** \(x^2 - 2x - 8\)

\(-1, 8; 1, -8; -2, 4; 2, -4\)
\(x^2 - 2x - 8 = (x - 2)(x + 4)\)

Find the pair whose sum is +2: 4 + -2 = 2

Make sure you take out the highest common factor to factorise fully.
The coefficient of $x^2$ is not always equal to 1.

**Example 5**

Factorise $4x^2 + 8x + 3$

$4x^2 + 8x + 3$

Factor pairs of 4 are $1, 4; -1, -4; 2, 2$ and $-2, -2$

Factor pairs of 3 are $1, 3$ and $-1, -3$

Discard the negative factors since all the coefficients in the equation are positive.

Use trial and error:

$(x + 1)(4x + 3) = 4x^2 + 7x + 3$  $\times$

$(2x + 1)(2x + 3) = 4x^2 + 8x + 3$  $\checkmark$

Hence $4x^2 + 8x + 3 = (2x + 1)(2x + 3)$

**Example 6**

Expressions in the form $x^2 - a^2$ are called the difference of two squares (DOTS).

You can factorise the difference of two squares expressions using $x^2 - a^2 = (x + a)(x - a)$

The difference of two squares can be shown by considering a square of length $x$ which contains a smaller square of length $a$.

Shaded area = $x^2 - a^2$

But also shaded area $= x(x - a) + a(x - a)$

$= (x + a)(x - a)$

Hence $x^2 - a^2 = (x + a)(x - a)$

**Example 7**

You can sometimes simplify algebraic fractions by first factorising.

Simplify the algebraic fraction $\frac{12x - 6x^2}{3x^2}$

Factorise the numerator:

$\frac{12x - 6x^2}{3x^2} = \frac{6x(2 - x)}{3x^2}$

There is a common factor of $3x$ on the top and the bottom.

$\frac{6x}{3x} \times \frac{2(2 - x)}{3x}$

Cancel down:

$\frac{2(2 - x)}{x}$

The simplified fraction is $\frac{2(2 - x)}{x}$

**Example 8**

Simplify the fraction $\frac{x^2 + 6x + 8}{x + 4}$

Factorise the numerator:

$x^2 + 6x + 8 = (x + 2)(x + 4)$

The fraction becomes

$\frac{(x + 2)(x + 4)}{(x + 4)}$

Cancel out the common factor $(x + 4)$:

$\frac{x^2 + 6x + 8}{x + 4} = x + 2$

In the next example you have to factorise both the numerator and the denominator to simplify the fraction.

**Example 9**

Simplify the fraction $\frac{2x^2 + 5x - 3}{2x^2 + 7x + 3}$

Factorise both the numerator and denominator:

$2x^2 + 5x - 3 = (2x - 1)(x + 3)$ and $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

The fraction becomes

$\frac{(2x - 1)(x + 3)}{(2x + 1)(x + 3)}$

Cancel down by the common factor $(x + 3)$:

$\frac{2x - 1}{2x + 1}$

The simplified fraction is $\frac{2x - 1}{2x + 1}$

You can only cancel by a term which is a factor of both the numerator and the denominator.
Exercise 0.1

1 Simplify these expressions.
   a $4a + 3b - 7a - 2$
   b $m^2 + 2m - n^2 - 2m^2$
   c $2ab + 5ab - 7ab + ab$
   d $-pq + 2p^2q^2 + p^2q^3 - pq$
   e $3 - 2a + 6 - b$
   f $x^2 - 4 + 4x^3 - 3$
   g $3(a - b) - 4(a + b)$
   h $2(1 - 3x) - 2(3 - x)$
   i $a(a - b) - b(a - b)$
   j $b(a - b) - a(b - a)$

2 Expand and simplify these expressions.
   a $(x + 3)(x + 5)$
   b $x(x - 4)$
   c $(x - 3)(x + 5)$
   d $(2x - 1)(x - 3)$
   e $(x^2 - 2)(x + 4)$
   f $(2 - x)(5 - 2x)$

3 Factorise these expressions.
   a $4 - 2x$
   b $3x^2 + 6x$
   c $x^2 + 6x + 9$
   d $x^2 - 7x + 12$
   e $x^2 + 8x - 9$
   f $x^2 - 4x$
   g $x^2 - 16$
   h $2x^2 - 18$
   i $x^2 - 5x - 36$
   j $x^2 + 22x - 48$

4 Factorise these expressions.
   a $2x^2 + 3x + 1$
   b $3x^2 + 5x + 2$
   c $3x^2 - 5x - 2$
   d $6x^2 + 5x + 1$
   e $6x^2 + 17x - 3$
   f $12x^2 - 11x + 2$

5 Simplify these fractions.
   a $\frac{2x^2 + x}{x}$
   b $\frac{x^2 - x^3}{x^2}$
   c $\frac{3x + 6x^2}{3x}$
   d $\frac{6x + 8x^2}{2x}$
   e $\frac{5x - 15x^2}{10}$
   f $\frac{2x^3 + 4x^2 + 6x}{2x}$
   g $\frac{4x^3 + 6x^2}{8x}$
   h $\frac{x^2 - x}{x - 1}$

6 Simplify these fractions by factorising where possible.
   a $\frac{x^2 + 3x + 2}{x + 2}$
   b $\frac{x^2 - 2x + 1}{x - 1}$
   c $\frac{x^2 - x - 6}{x + 2}$
   d $\frac{x^2 - 2x - 8}{x - 4}$
   e $\frac{x^2 + 3x^2 + 2x}{x^2 + 3x + 2}$
   f $\frac{4x^2 - 4x + 1}{2x - 1}$
   g $\frac{6x^2 - x - 1}{2x^2 + x - 1}$
   h $\frac{x^2 + 7x + 10}{2x^2 + 11x + 5}$
   i $\frac{2x^2 - 9x + 9}{2x^2 - 11x + 12}$
A formula is used to express the relationship between two or more variables.

The formula $A = \frac{1}{2}bh$ expresses the relationship between $A$, the area, $b$, the base and $h$, the height of a triangle.

By substituting values into a formula you can work out the value of an unknown.

**Formulae**

Given the values $p = 3$, $q = -5$, $r = 2$ find the value of $T$ in each case.

- $T = \frac{p^2 - q}{r}$
  - Substitute in the values: $T = \frac{(3)^2 - (-5)}{2}$
  - Simplify: $T = \frac{9 + 5}{2} = \frac{14}{2} = 7$

- $T = \frac{3p}{r} - \frac{2q}{q}$
  - Substitute in the values: $T = \frac{3 \times 3}{2} - \frac{2 \times 2}{-5}$
  - Simplify by expressing the fractions with a common denominator:
    - $T = \frac{9 + 4}{10}$
    - $T = \frac{13}{10}$
    - $T = 1.3$

Sometimes you will need to change the subject of a formula by rearranging its terms.

**Example 1**

Make $x$ the subject of the formula $xyz + t = M$.

- Isolate the $x$-term: $xyz = M - t$
- Divide both sides of the formula by $yz$: $x = \frac{M - t}{yz}$

**Example 2**

Make $y$ the subject of the formula $hx + p = 1$.

- Isolate the $y$-term: $y = \frac{1 - hx}{p}$
- Multiply both sides by $-1$ and rearrange: $y = \frac{p}{-h} - \frac{x}{p}$
- Invert both fractions: $y = \frac{p}{-h} - \frac{x}{p}$
- Multiply both sides by $p$: $p \times y = p \times \frac{p}{-h} - \frac{x}{p}$
- Simplify: $y = \frac{px}{h} - \frac{x}{p}$

**Exercise 0.2**

1. Given the values $a = 4$, $b = 9$ and $c = 3$, find the value of $P$ in each case.
   - $a \quad P = 3a^2 - bc$
   - $b \quad P = a - b + c$
   - $c \quad P = \sqrt[3]{(a^2 + c^2)}$
   - $d \quad P = \sqrt[3]{a} \sqrt[3]{b}$
   - $e \quad P = (a + b)(b - c)$
   - $f \quad P = \frac{2a + 2b}{c}$
   - $g \quad P = \sqrt[3]{b} \sqrt[3]{c}$
   - $h \quad P = \frac{bc}{a}$
   - $i \quad P = \frac{b - a}{c}$
   - $j \quad P = \left(\frac{b + c}{a}\right)^2$
2 Use substitution to find the value of the subject of each formula.
   a \( A = \pi r^2; \) \( \pi = 3, \ r = 0.4 \)
   b \( A = 4\pi r^2; \) \( r = \frac{1}{2}, \ \pi = 3 \)
   c \( L = 2(a + b); \) \( a = 0.07, \ b = 0.7 \)
   d \( I = \frac{PF}{100}; \) \( P = 160, \ T = 3, \ R = 5 \)
   e \( V = \pi r^3; \) \( r = 4, \ \pi = 3 \)
   f \( l = \sqrt{(a^2 - b^2)}; \) \( a = 13, \ b = 12 \)
   g \( A = \pi l^2; \) \( r = 0.2, \ l = 1.4, \ \pi = 3 \)
   h \( V = \frac{4}{3}\pi r^3; \) \( \pi = 3, \ r = 3 \)
   i \( T = 2\pi \sqrt{\frac{l}{g}}; \) \( \pi = 3, \ g = 32, \ l = 2 \)
   j \( f = \frac{uv}{u + v}; \) \( u = \frac{1}{2}, \ v = \frac{1}{3} \)

3 Make \( x \) the subject of each formula.
   a \( y = 3x - 4 \)
   b \( 2y + x - 3 = 0 \)
   c \( y = mx + c \)
   d \( 3(x + y) = 5 \)
   e \( ax + by + c = 0 \)
   f \( (x - 1)(y + 2) = 3 \)
   g \( \frac{1}{x} + \frac{1}{y} = 1 \)
   h \( 3x^2 + y - 4 = 0 \)
   i \( \frac{2}{x + 1} = y \)
   j \( \frac{1}{x + 1} - \frac{1}{y - 1} = 1 \)

4 Find the value of the unknown in each formula.
   a \( 4a - 3b + 2c = 0; \) \( a = 3, \ b = 5; \) find \( c \)
   b \( x^2 + y^2 = r^2; \) \( x = 16, \ r = 20; \) find \( y \)
   c \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r}; \) \( r = 5, \ p = 6; \) find \( q \)
   d \( \frac{2m - n}{4m} = \frac{1}{p}; \) \( m = 2, \ n = 3; \) find \( p \)
   e \( (2 + p)(3 + q) = r; \) \( p = \frac{1}{4}, \ r = 2; \) find \( q \)
   f \( \frac{2m - c}{3c} = \frac{1}{g}; \) \( a = 2, \ c = 4; \) find \( b \)
   g \( \frac{4}{3}m(n - 1) = \frac{1}{p}; \) \( m = 0.5, \ p = 1.5; \) find \( n \)
   h \( \frac{4 - x}{3 - x} = \frac{1}{s}; \) \( s = -3; \) find \( r \)

5 a In vertical motion under gravity the distance, \( s, \) which a stone falls is related to its initial speed, \( u, \) time, \( t, \) and acceleration, \( a, \) due to gravity where
   \[ s = ut + \frac{1}{2}at^2 \]
   Make \( u \) the subject of the formula.
   Find the value of \( u \) when \( s = 200, \ a = 10 \) and \( t = 5. \)
   b A force acts on a particle of mass, \( m, \) and the velocity of the particle changes from \( u \) to a new velocity, \( v. \)
   The impulse of the force is measured by the formula
   \[ I = mv - mu \]
   Make \( m \) the subject of the formula.
   Find the value of \( m \) when \( I = 240, \ u = 0.6 \) and \( v = 1.8. \)
0.3 Solving linear equations

Linear equations involve a variable, such as \( x \).
You can solve a linear equation to find a unique solution.

**Example 1**
Find the value of \( x \) where \( 3x - 1 = 8 \)

1. Isolate the \( x \)-term: \( 3x = 8 + 1 \)
2. Simplify: \( 3x = 9 \)
3. Divide both sides by 3: \( \frac{3x}{3} = \frac{9}{3} \)
4. Hence \( x = 3 \)

You want to get \( x \) on its own.

Check your answer by substituting for \( x \) in the original equation.

Sometimes you will need to simplify an equation before solving it.

**Example 2**
Solve the equation \( 5x - 2 = 3x + 4 \)

1. Collect the \( x \)-terms on one side and the constants on the other side: \( 5x - 3x = 4 + 2 \)
2. Collect like terms: \( 2x = 6 \)
3. Divide both sides by 2: \( \frac{2x}{2} = \frac{6}{2} \)
4. Hence \( x = 3 \)

You want to get \( x \) on its own.

Check your answer by substituting for \( x \) in the original equation.

**Example 3**
Solve the equation \( 2(4r - 1) - 3(r - 2) = 0 \)

1. Expand the brackets: \( 8r - 2 - 3r + 6 = 0 \)
2. Collect like terms: \( 5r + 4 = 0 \)
3. Rearrange: \( 5r = -4 \)
4. Divide both sides by 5: \( \frac{5r}{5} = \frac{-4}{5} \)
5. Hence \( r = -\frac{4}{5} \)

Take care when multiplying negative terms.

Check your answer by substituting for \( r \) in the original equation.

Linear equations may involve algebraic fractions.

**Example 4**
Solve the equation \( \frac{x + 1}{3} + 1 = \frac{x}{2} \)

1. Combine the terms on the LHS: \( \frac{x + 1}{3} + \frac{3}{3} = \frac{x}{2} \)
2. Multiply both sides by 6: \( 2(x + 4) = 3x \)
3. Expand: \( 2x + 8 = 3x \)
4. Collect like terms and rearrange: \( -x = -8 \)
5. Divide both sides by -1: \( x = 8 \)

6 is the LCM of 2 and 3.

Check your answer by substituting \( x = 8 \) into the original equation.

**Exercise 0.3**
1. Find the value of \( x \) for each equation.
   - a \( 2x + 5 = 9 \)
   - b \( 3 - 4x = -9 \)
   - c \( 7 = 2 - 2x \)
   - d \( \frac{1}{2}x + 1 = 3 \frac{1}{2} \)
   - e \( \frac{2x}{3} - 2 = 4 \frac{3}{3} \)
   - f \( 1 + \frac{x}{3} = -\frac{2}{3} \)
   - g \( 3x - 2 = x + 6 \)
   - h \( 2 - x = 4x - 8 \)
   - i \( 5x - 4 = 2x - 16 \)
   - j \( \frac{3}{2}x + \frac{3}{2} = 1 \)
   - k \( 3x + 2 + 7x - 4 = -4x \)
   - l \( \frac{4x}{3} = -2 + \frac{2x}{3} \)

2. Find the value of the unknown in each equation.
   - a \( 3(t - 2) = 6 \)
   - b \( 5(2r + 1) = -15 \)
   - c \( 2(1 - 3p) = 0 \)
   - d \( 1 = 3(y - 1) \)
   - e \( -2(r - 3) = 4 \)
   - f \( 3 - 4(x - 1) = 0 \)
   - g \( 3(y + 2) = 2(y - 1) \)
   - h \( 2(2r - 1) - 3(t + 1) = 0 \)
   - i \( 4 - 2(t + 3) = 0 \)
   - j \( 4(1 - r) - 3(r - 1) = 0 \)
   - k \( 5(2x - 1) = -2(5x + 1) \)
   - l \( \frac{1}{2}(x + 3) - (1 - x) = 1 \)

3. Solve these equations.
   - a \( \frac{x}{3} - \frac{2x}{5} = 2 \)
   - b \( \frac{x + 1}{2} = \frac{2x}{5} \)
   - c \( \frac{5}{x + 1} = \frac{2}{x - 1} \)
   - d \( 3 + \frac{2 - x}{2} = \frac{2x + 1}{3} \)
   - e \( \frac{3x}{x} = \frac{x}{x + 1} \)
   - f \( \frac{x + 2}{x} + \frac{2x - 3}{3} = 2 \)
   - g \( \frac{x + 1}{3x} = \frac{x + 2}{3x + 1} \)
   - h \( \frac{x(1 + 2x)}{x - 1} = 2x - 1 \)
0.4 Plotting graphs

Linear functions

Linear functions can be written in the form
\[ y = mx + c \]
where \( m \) and \( c \) are constants.

A linear function is represented graphically by a straight line. \( m \) is the gradient and \( c \) is the \( y \)-intercept of the graph.

Here are some examples of linear functions.

\[ y = 2x + 3 \]
\[ 3x - 2y + 1 = 0 \]
\[ 4y - x = 3 \]

gradient = 2
gradient = \frac{3}{2}
gradient = \frac{1}{4}

\[ y \text{-intercept} = 3 \]
\[ y \text{-intercept} = \frac{1}{2} \]
\[ y \text{-intercept} = \frac{3}{4} \]

To find the \( y \)-axis crossing, substitute \( x = 0 \) into the linear equation and solve for \( y \).
To find the \( x \)-axis crossing, substitute \( y = 0 \) into the linear equation and solve for \( x \).

Draw the graph of \( y = 2x + 1 \)

Make a table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

Use your table of values to draw the straight line graph.

Check with the equation:
gradient = 2; \( y \)-intercept = 1

EXAMPLE 2

Plot the graph of \( y = -\frac{1}{2}x + 2 \)

Make a table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Use your table of values to draw the straight line graph.

Check with the equation:
gradient = -\frac{1}{2}; \( y \)-intercept = 2

Notice that the line graphs in Examples 1 and 2 slope in different directions.

The direction of slope of a straight line relates to the sign of the gradient.

positive gradient

negative gradient

Sometimes you will need to rearrange the equation into the form \( y = mx + c \) before you can plot the line graph.
Quadratic functions

The general form of a quadratic function is

\[ y = ax^2 + bx + c \]

where \( a, b \) and \( c \) are constant values and \( a \neq 0 \).

The graph of a quadratic function is in the shape of a Y. You can draw the graph of a quadratic function by constructing a table of values. Use suitable values of \( x \) to include the important features.

Example 1

Plot the graph of the function \( y = x^2 + 1 \) taking values of \( x \) from -3 to 3.

Make a table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Use your table of values to draw the graph:

The curve intersects the \( y \)-axis when \( x = 0 \).

When \( x = 0, y = 1 \) so the coordinates of this point are (0, 1).

Example 3

Function notation is often used to describe an equation.

Plot the graphs of \( y = f(x) \) and \( y = -f(x) \) where \( f(x) = -2x \)

Let \( y = f(x) \), then \( y = -2x \)

\( y = -f(x) \) gives \( y = 2x \)

Make a table of values for \( y = f(x) = -2x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Make a table of values for \( y = -f(x) = 2x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Use your tables of values to draw the straight line graphs:

Gradient of \( f(x) \) is \(-2\).

Gradient of \(-f(x)\) is \(2\).
Plot the graph of the function \( f(x) = x^2 - 6x + 8 \) taking values of \( x \) from 0 to 5. Find the coordinates of the points where the curve intersects the \( x \)-axis and the \( y \)-axis. Estimate the gradient of the tangent at the point where \( x = 1 \).

Construct a table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
<tr>
<td>(-6x )</td>
<td>0</td>
<td>-6</td>
<td>-12</td>
<td>-18</td>
<td>-24</td>
<td>-30</td>
</tr>
<tr>
<td>(+8 )</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>( y )</td>
<td>8</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Use your table of values to plot the graph:

\( f(0) = 8 \) so the curve intersects the \( y \)-axis at (0, 8).
\( f(2) = 0 \) and \( f(4) = 0 \) so the curve intersects the \( x \)-axis at (2, 0) and (4, 0).

Exercise 0.4

1. Plot the graph of each function taking the given values of \( x \).
   a. \( y = x - 3 \), \( x = -2 \) to 4
   b. \( y = -x + 4 \), \( x = -2 \) to 5
   c. \( y = 2x - 3 \), \( x = -1 \) to 5
   d. \( y = -3x + 5 \), \( x = -2 \) to 3

2. Plot the graph of each function taking \( x \)-values from -2 to 4.
   a. \( f(x) = \frac{1}{2}x^2 - 2 \)
   b. \( f(x) = -2x + 4 \)
   c. \( f(x) = -x + 2.5 \)
   d. \( f(x) = 5x - 10 \)

3. Make \( y \) the subject of each equation and plot the graphs taking suitable values of \( x \).
   a. \( 2y - 3x + 2 = 0 \)
   b. \( x + 2y - 8 = 0 \)
   c. \( 5 = 4x - 2y \)

4. Plot the graph of each equation stating the value of the gradient.
   a. \( \frac{1}{2}(y + 1) = \frac{1}{2}(x - 1) \)
   b. \( y = 2f(x) \), where \( f(x) = 1 - x \)
   c. \( y = -2 \)

5. a. Copy and complete the table of values for \( y = x^2 - 4x \) for \( x = -1 \) to \( x = 5 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-4x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Write down the coordinates of the points where the curve intersects the
   i. \( x \)-axis
   ii. \( y \)-axis.
6 a Copy and complete the table of values for \( y = x^2 - 2x - 8 \) for \( x = -3 \) to \( x = 5 \).

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(-2x)</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-8)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Plot the graph of the quadratic.

c Find the coordinates of the points where the curve intersects the
i \( x \)-axis
ii \( y \)-axis.

7 a Copy and complete the table of values for \( y = 2 + x - x^2 \) for \( x = -2 \) to \( x = 3 \).

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x^2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(+2)</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Plot the curve.

c Find the coordinates of the points where the curve intersects the
i \( x \)-axis
ii \( y \)-axis.

8 a Plot the graphs of \( y = x^2 + 2 \) and \( y = x^2 - 3 \) on the same axes taking values of \( x \) from -3 to 3.

b What do you notice about the relative positions of these curves?

9 a Plot the graphs of \( y = -x^2 + 3 \) and \( y = -x^2 - 4 \) on the same axes taking values of \( x \) from -3 to 3.

b What do you notice about the relative positions of these curves?

c Compare these curves to the ones drawn in question 8.

10 a Plot the graphs of \( y = x^3 \), \( y = 2x^2 \) and \( y = \frac{1}{2}x^3 \) on the same axes for \( x = -3 \) to \( x = 3 \).

b What similarities or differences do you notice between the curves?

11 a Plot the curve of \( y = 3x^2 + 7x - 20 \) taking values of \( x \) from -5 to 3.

b Write down the coordinates of the points where the curve intersects the
i \( x \)-axis
ii \( y \)-axis.

c Use the graph to estimate the coordinates of its turning point.

d Estimate the gradient of the tangent at the point where \( x = -2 \).

12 a Plot the graph of the function \( f(x) = x^2 - x - 6 \) taking values of \( x \) from -3 to 4.

b Find the values of \( x \) for which \( f(x) = 0 \)

c Plot the graph of \( g(x) = x - 5 \) on the same axes.

d Estimate the values for which \( f(x) = g(x) \) giving your values of \( x \) to 1 d.p.

How many values are there?

This is the point at which the graph changes direction.
The midpoint and length of a line segment

The midpoint of the straight line joining two points \((x_1, y_1)\) and \((x_2, y_2)\) has coordinates \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

Example 1
Points \(A\) and \(B\) have coordinates \((1, 4)\) and \((-2, -3)\) respectively.
Find the midpoint of the straight line \(AB\).

Use the formula to find the coordinates of the midpoint:

\[
\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + (-2)}{2}, \frac{4 + (-3)}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)
\]

You can sketch a diagram to check if your answer is sensible.

Example 2
The midpoint of the straight line joining two points \((x_1, y_1)\) and \((x_2, y_2)\) has coordinates \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

Exercise 0.5
1. The diagram shows three points \(A\), \(B\) and \(C\).

   - \(A\) \((2, 4)\)
   - \(B\) \((5, 5)\)
   - \(C\) \((2, 4)\)

   Find
   a. the midpoint of the line joining the points \(A\) and \(B\)
   b. the length of the line joining the points \(A\) and \(C\)
   c. the midpoint of the line joining points \(B\) and \(C\).

   Show your working in each case.

2. Find
   i. the midpoint
   ii. the length of the line joining each pair of points.

   a. \((0, 0)\) and \((2, 3)\)
   b. \((1, 1)\) and \((4, 2)\)
   c. \((-1, -1)\) and \((6, 5)\)
   d. \((0, 2)\) and \((4, 1)\)
   e. \((-2, 4)\) and \((1, 1)\)
   f. \((-2, 0)\) and \((1, 3)\)

3. A line which joins two points \(A\) and \(B\) has a midpoint with coordinates \((\frac{7}{2}, \frac{3}{2})\).

   Use a suitable formula to find the coordinates of point \(B\) if
   a. point \(A\) is at \((1, 1)\)
   b. point \(A\) is at \((3, -1)\).

4. Two points \(A\) and \(B\) are joined by a line.

   Point \(A\) has coordinates \((-1, -1)\).

   The midpoint of the line \(AB\) has coordinates \((1, 2)\).

   Show that the length of line \(AB\) is \(2\sqrt{13}\).
0.6 Simultaneous equations

You can solve a pair of simultaneous equations by eliminating one of the variables.

The solution gives the values of the two variables which satisfy both equations at the same time.

Solve the simultaneous equations for \( x \) and \( y \):

\[
2x - 3y = 13, \quad 5x + 3y = 1
\]

Aim to eliminate either \( x \) or \( y \).

Label the equations and add them:

\[
2x - 3y = 13 \quad (1) \\
\quad + \\
5x + 3y = 1 \quad (2)
\]

Collect like terms:

\[
2x + 5x = 13 + 1
\]

Divide by 7:

\[
x = 2
\]

Substitute \( x = 2 \) into equation (1) to find the value of \( y \):

\[
2(2) - 3y = 13
\]

Rearrange and simplify:

\[
2 \cdot 2 - 3y = 13
\]

Divide by -3:

\[
y = -3
\]

Hence the solution is \( x = 2 \) and \( y = -3 \).

Sometimes you will need to multiply one of the equations by a constant to make the coefficients of either \( x \) or \( y \) the same.

EXAMPLE 2

Find \( x \) and \( y \) when \( 3x + y = -5 \) and \( 4x - 3y = -11 \)

Rewrite and label the equations:

\[
3x + y = -5 \quad (1) \\
4x - 3y = -11 \quad (2)
\]

Multiply equation (1) by 3 to make the coefficients of \( y \) the same, but with opposite signs. Label this equation (3).

\[
3 \times (1): \\
9x + 3y = -15 \quad (3)
\]

Compare with equation (2):

\[
4x - 3y = -11 \quad (2)
\]

Add equations (2) and (3):

\[
9x + 4x = -15 - 11
\]

Simplify:

\[
13x = -26
\]

Divide by 13:

\[
x = -2
\]

Substitute \( x = -2 \) into equation (1) to find the value of \( y \):

\[
3(-2) + y = -5
\]

Rearrange:

\[
y = -5 + 6
\]

Hence the solution is \( x = -2 \) and \( y = 1 \).

Sometimes both equations need to be multiplied by a constant so that one variable can be eliminated.

It helps to write one equation under the other so that you can compare the \( x \)- and \( y \)-coefficients.

Remember to multiply all of the terms by 3.

Check by substituting the values of \( x \) and \( y \) into the original equations.
Solve the simultaneous equations

\[ 3y + 4x = -4 \quad \text{and} \quad 2y + 6x = -1 \]

**Rewrite and label the equations:**

\[ 3y + 4x = -4 \quad (1) \]
\[ 2y + 6x = -1 \quad (2) \]

**Multiply equation (1) by 2 and equation (2) by 3:**

\[ 3y + 4x = -8 \quad (3) \]
\[ 2y + 6x = -3 \quad (4) \]

**Equation (3) – equation (4):**

\[ 0y + 8x = -8 + 3 \]
\[ -10x = -5 \]
\[ x = \frac{1}{2} \]

**Substitute \( x = \frac{1}{2} \) into equation (1):**

\[ 3y + 4 \left( \frac{1}{2} \right) = -4 \]
\[ 3y + 2 = -4 \]
\[ 3y = -2 \]
\[ y = -\frac{2}{3} \]

**Hence the solution is** \( x = \frac{1}{2} \) **and** \( y = -\frac{2}{3} \).

You can solve a pair of simultaneous equations by substitution.

---

**EXAMPLE 4**

Solve by substitution \( y = 3x + 4 \), \( 4x + y = -3 \)

**Label the equations:**

\[ y = 3x + 4 \quad (1) \]
\[ 4x + y = -3 \quad (2) \]

**Substitute** \( y = 3x + 4 \) **in equation (2):**

\[ 4x + (3x + 4) = -3 \]
\[ 7x + 4 = -3 \]
\[ x = -1 \]

**Rearrange:**

\[ 7x = -3 - 4 \]
\[ x = -1 \]

**Substitute** \( x = -1 \) **into equation (1) to find the value of** \( y \):

\[ y = 3x + 4 \]
\[ y = -3 + 4 = 1 \]

The solution is \( x = -1 \) **and** \( y = 1 \).

You can also estimate the solution to a pair of simultaneous equations by plotting their graphs.

---

**EXAMPLE 5**

Solve graphically \( 2x + y = 7 \), \( y = x + 1 \)

**Make a table of values for each equation:**

\[
\begin{array}{c|c|c}
\text{x} & \text{y} = x + 1 & \text{2x + y = 7} \\
\hline
0 & 2 & \text{Graph}\text{a} \\
1 & 3 & \text{Graph}\text{b} \\
2 & 4 & \text{Graph}\text{c} \\
3 & 5 & \text{Graph}\text{d} \\
\end{array}
\]

**Use your tables to draw the graphs on the same pair of axes.**

The point of intersection has coordinates \((2, 3)\) – this gives the solution \( x = 2 \) and \( y = 3 \).
Exercise 0.6

1. In each pair of simultaneous equations eliminate one of the unknowns by adding or subtracting the equations. Hence find the values of $x$ and $y$.

   - $a \quad x + 2y = 3$
   - $x + y = 2$
   - $c \quad x + 2y = 5$
   - $3x + 2y = 3$
   - $e \quad 2y - 3x = 12$
   - $x - 2y = -8$
   - $g \quad 2y - x = 8$
   - $3x - 2y = -14$
   - $i \quad 4x - 3y = 1$
   - $4x - 5y = 7$

   - $b \quad 2x - y = 5$
   - $x + y = 1$
   - $d \quad 2x - 4y = 6$
   - $2x + y = \frac{13}{2}$
   - $f \quad 4x - y = 3$
   - $2y - 4x = -4$
   - $h \quad 3x - y = 7$
   - $y - 2x = -3$
   - $j \quad 2x + y = 5$
   - $4x + y = 12$

2. In each pair of simultaneous equations multiply one equation by a suitable constant then add or subtract the equations to eliminate an unknown. Hence find the values of $x$ and $y$.

   - $a \quad x + 3y = 7$
   - $2x - y = 0$
   - $c \quad 3y + 2x = -1$
   - $y - 3x = -15$
   - $e \quad 4x - y = 4$
   - $2x - 2y = 5$
   - $g \quad 2x + y = 6$
   - $5x + 3y = 16 \frac{1}{2}$
   - $i \quad 2y - 3x = 8$
   - $2x - y = -5 \frac{1}{2}$

   - $b \quad 2x - y = 5$
   - $3x - 2y = 8$
   - $d \quad 3x + 2y = -10$
   - $2x - 4y = 4$
   - $f \quad 3y + 4x = 15$
   - $2y - x = 10$
   - $h \quad 4x - 3y = -7$
   - $12x - y = -5$
   - $j \quad 6x + 3y = 3$
   - $4y + 2x = -20$

3. Use the method of substitution to solve these pairs of equations.

   - $a \quad x + 2y = 8$
   - $2x + 3y = 14$
   - $c \quad 4x + y = 14$
   - $x + 3y = 13$
   - $e \quad 3p - q = 5$
   - $2p + 5q = 7$

   - $b \quad x + 2y = 1$
   - $2x + 3y = 4$
   - $d \quad c + d = 4$
   - $2c + d = 5$
   - $f \quad 2y = 4 + x$
   - $6x - 5y = 18$

4. Plot the graphs of each of these simultaneous equations on the same diagram and solve them for $x$ and $y$.

   - $a \quad y = 2x + 1$
   - $y = -2x + 1$
   - $c \quad y = \frac{1}{2}x + 1$
   - $y = 2x - 2$
   - $d \quad y = x - 4$
   - $y = 2x - 3$

5. Estimate solutions to these simultaneous equations using a graphical method.

   - $a \quad 2y - x = -1$
   - $y - x = 2$
   - $c \quad 2y = 3x - 2$
   - $y + 2x = 4$
   - $b \quad 3y + 4 = 3x$
   - $y + x = 2$

   - $d \quad 2y + 1 = 5x$
   - $3x + 2y = 7$

6. Solve these problems by forming and solving simultaneous equations.

   - $a \quad$ When a number $x$ is subtracted from twice the number $y$ the answer is 2.
     
     Three times $y$ subtracted from four times $x$ is 17.
     
     Set up two equations and solve simultaneously to find $x$ and $y$.
     
     $b \quad$ Two packs of sandwiches and a cup of coffee cost £4.20 at Jean’s Café whilst three packs of sandwiches and two cups of coffee cost £6.70.
     
     Find, by setting up two simultaneous equations, the cost of a pack of sandwiches and the cost of a cup of coffee.
     
     $c \quad$ At a theme park the entrance fees are £2.50 for adults and £1.00 for children.
     
     A family of two adults and two children pay £74 and a party of three adults and five children pay £141. Find the admission charge per person.
0.7 Inequalities

The four symbols used to describe an inequality are

\(<\), \(\text{\(\geq\)}\), \(\text{\(\leq\)}\), and \(\text{\(\geq\)}\).

\(n < 2\) means the number \(n\) is less than 2
\(n > 2\) means the number \(n\) is greater than 2
\(n \leq 2\) means the number \(n\) is less than or equal to 2
\(n \geq 2\) means the number \(n\) is greater than or equal to 2

Inequalities can be shown on a number line.

\[-3\] \(-2\] \(-1\] \(0\] \(1\] \(2\] \(3\] \(4\] \(5\]

\(n < 2\) \(\ boyfriend
\(n > 2\) \(\ boyfriend
\(n \leq 2\) \(\ boyfriend
\(n \geq 2\) \(\ boyfriend

You can use a combination of inequality signs to define a set of values.

**Example 1**

On a number line show the set of values satisfied by the inequality \(-1 < n \leq 3\).

\[-3\] \(-2\] \(-1\] \(0\] \(1\] \(2\] \(3\] \(4\] \(5\]

\(n < 2\) \(\ boyfriend
\(n > 2\) \(\ boyfriend
\(n \leq 2\) \(\ boyfriend
\(n \geq 2\) \(\ boyfriend

You can combine two different inequalities to define a set of solutions.

**Example 2**

On a number line show the solution set satisfied by both \(n < 4\) and \(n \geq -2\).

\[-3\] \(-2\] \(-1\] \(0\] \(1\] \(2\] \(3\] \(4\] \(5\]

\(n < 4\) \(\ boyfriend
\(n \geq -2\) \(\ boyfriend

\(n\) is strictly less than 4 and greater than or equal to -2. Show each inequality on a number line and then combine to give the final set of solutions.

\(n\) \(-2\] \(-1\] \(0\] \(1\] \(2\] \(3\] \(4\] \(5\]

Sometimes you will need to simplify an inequality before you can show the set of solutions on a number line.

Simplify \(-4 < 3x + 2 \leq 11\) and show the solution set on a number line.

\(-4 < 3x + 2 \leq 11\)

The inequality has two parts. Solve them separately.

**Solve the LH part:**

\(-4 < 3x + 2\)

Subtract 2 from each part:

\(-4 - 2 < 3x + 2 - 2\)

Simplify:

\(-6 < 3x\)

Divide by 3:

\(-2 < x\)

**Solve the RH part:**

\(3x + 2 \leq 11\)

\(3x + 2 - 2 \leq 11 - 2\)

\(3x \leq 9\)

\(x \leq 3\)

\(-2 < x \leq 3\)

Represent the solution \(-2 < x \leq 3\) on a number line:

\[-3\] \(-2\] \(-1\] \(0\] \(1\] \(2\] \(3\] \(4\] \(5\]

\(-2 < x \leq 3\)

Solve each part separately.

Remember that dividing or multiplying by a negative number reverses the sign of the inequality.

Sometimes you will need to simplify an inequality before you can show the set of solutions on a number line.
You can show the solution set of an inequality involving two variables as a region on a graph.

- Use shading to show where the solutions lie.
- Use continuous lines to represent the boundary of \( \leq \) or \( \geq \) and dashed lines to represent the boundary of \( < \) or \( > \).

**Example 5**

Shade the region on a graph to show the values which satisfy \( x > 2 \) and \( y \leq 3 \).

**Exercise 0.7**

1. Use a number line to show the solutions of these inequalities.
   
   - a. \( n < 3 \)  
   - b. \( n \geq -1 \)  
   - c. \( n > 0 \) and \( n < 4 \)  
   - d. \( n \geq -2 \), \( n < 2 \)  
   - e. \( n < -1 \), \( n \geq -3 \)  
   - f. \( -1 < n \leq 3 \)  
   - g. \( -4 < n < -1 \)  
   - h. \( 3 < n \leq 4 \)  
   - i. \( -1 \leq n \leq 5 \)  
   - j. \( -4 \leq n < -2 \)  

2. Simplify the inequalities and show the solutions as a range on a number line.
   
   - a. \(-3 < x + 1 < 4\)  
   - b. \(-2 < x - 2 \leq 0\)  
   - c. \(2 \leq 2x < 8\)  
   - d. \(-5 \leq 2x + 1 < 7\)  
   - e. \(0 < 2x - 4 \leq 3\)  
   - f. \(-11 < 3x - 5 < 4\)  
   - g. \(-1 < 3 + 2x < 5\)  
   - h. \(9 < 4x - 1 \leq 21\)  

3. For each pair of inequalities, shade the required region on a graph.
   You may need to simplify the inequalities first.
   
   - a. \( x > 1 \), \( y > 2 \)  
   - b. \( x \leq 2 \), \( y > 0 \)  
   - c. \( x > -1 \), \( y \leq 3 \)  
   - d. \( x \geq 0 \), \( y \geq 2 \)  
   - e. \( x + 2 > 0 \), \( y - 1 > 0 \)  
   - f. \( 2x \leq 5 \), \( 3y > 6 \)  
   - g. \( 2x + 1 < 0 \), \( 3y - 1 > 0 \)  
   - h. \( 4x - 3 > 9 \), \( 5 + 2y \leq 3 \)  
   - i. \( 3 + 2x \leq 1 \), \( 3 + 2y \leq -2 \)  
   - j. \( 2 + 3x < x, 5 + 2y > y \)  
   - k. \( 5x - 4 > 3x, 2(y - 1) < 3 \)  
   - l. \( 2(x + 3) < 10, 3(y - 1) > 2y \)
0.8 Circle theorems

**Theorem:** The angle between a tangent to a point on a circle and a radius at the same point is a right angle.

If the centre of the circle is at the point O and A is a point on the circumference of the circle, the tangent drawn to the circle at point A makes an angle of 90° with the radius, OA.

Angle OAT = 90°

**Theorem:** Tangents drawn from two points on the circumference of a circle intersect at a point and are of the same length.

Let T be the point of intersection of the tangents drawn from the points A and B on the circumference of a circle.

Angle OAT = angle OBT = 90°
Length OA = OB (radii)
Length OT is common to both triangles.

Triangles OAT and OBT are congruent.

\[ \therefore AT = BT \]

**Example 1**

The line AOB is a diameter of the circle where A and B are points on the circumference. C is another point on the circumference.

Angle ACB = 90°

The ends of the chord AB lie on the circumference of a circle. The chord divides the circle into a major segment (the larger one) and a minor segment.

The two angles ACB and ADB are in the same segment.

Angle ACB = Angle ADB

**Example 2**

AT and BT are tangents to the circle, centre O. Angle ATB = 45°. Find the obtuse angle AOB.

Angle OAT = OBT (= 90°)
(radius and tangent)
Angle sum of a quadrilateral is 360°.

Obtuse angle AOB = 360° – (90° + 90° + 45°)
= 135°

**Theorem:** The angle in a semicircle is a right angle.

The line AOB is a diameter of the circle where A and B are points on the circumference. C is another point on the circumference.

Angle ACB = 90°

Angle ACB is called ‘the angle in a semicircle’.

The points A, B, C and D lie on the circumference of a circle. The lines AC and BD intersect at E. Angle ADB = 38° and CBD = 29°. Find angles CAD and AED.

Draw the chord from C to D making two segments:

Angle CAD = angle CBD (angles on the same chord in the same segment)

\[ \therefore \text{angle CAD} = 29° \]

In triangle AED
Angle AED = 180° – (29° + 38°) (angle sum of triangle)
Angle AED = 113°
Prove that the opposite angles of a cyclic quadrilateral are supplementary.

Let the angle at $B$ be $p$ and the angle at $D$ be $q$.

Using the theorem that the angle at the centre is twice the angle at the circumference, obtuse angle $AOC = 2q$
reflex angle $AOC = 2p$
The sum of these two angles is $360°$
(angles at a point).
Hence $2p + 2q = 360°$
\[ p + q = 180° \]
So opposite angles of a cyclic quadrilateral are supplementary.

In a cyclic quadrilateral the four vertices lie on the circumference of a circle.

**Exercise 0.8**
The diagrams in this Exercise are not drawn to scale.

1. In these diagrams, O is the centre of the circle.
   Find the missing angles.

   a
   b
   c
   d
   e
   f $AB$ is a tangent.

Supplementary angles add up to $180°$. Using chord $DB$ you can prove that angles $A + C = 180°$.
2 In these diagrams, O is the centre of the circle. Find the missing angles.

a

b

A′ and B′ are tangents.

C

A

B

T

3 a Points A, B and C are on the circumference of the circle, centre O. BT is a tangent to the circle at point B. If the angle CBT = 59°, find the size of the angle COB.

b The points A, B, C and D are on the circumference of a circle.

C

A

B

T

Chord AB is parallel to chord CD.

Angle ACD = 42°.

AC and BD intersect at E.

Find angles

i DBA

ii BDC

iii BEC.

c The points A, B, C and D are on the circumference of a circle.

A′ is a tangent.

PAT is a tangent to the circle at point A.

AD = CD.

Angle EMT = 33°.

Find the angles

i ACD

ii CDA

iii CBA

iv CAP.

d The points A, B, C and D lie on the circumference of a circle.

AB = AD and BC = CD.

BA is produced to point E and angle EAD = 112°.

Find the sizes of the angles

i BAD

ii ABC

iii BCD

iv CDA.

4 a Prove that angles drawn on the same chord in the same segment of a circle are equal.

b Prove that the angle at the centre of a circle is twice the angle at the circumference when the angles are subtended by the same chord.

c Prove the alternate segment theorem.
For the right-angled triangle ABC the three trigonometric ratios are

\[ \sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC} \]

\[ \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC} \]

\[ \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB} \]

Pythagoras’ theorem states that

\[ AB^2 + BC^2 = AC^2 \]

In the right-angled triangle ABC, side \( AC = 9 \text{ cm} \) and angle \( A = 34^\circ \). Find the lengths of sides \( AB \) and \( BC \).

Use the cosine ratio to find \( AB \):

\[ \cos 34^\circ = \frac{AB}{9} \]

\[ 9 \times 0.8290 = AB \]

\[ AB = 7.46 \text{ cm} \]

Use the sin ratio to find \( BC \):

\[ \sin 34^\circ = \frac{BC}{9} \]

\[ 9 \times 0.5592 = BC \]

\[ BC = 5.03 \text{ cm} \]

The trigonometric ratios may also be applied to more practical questions. A diagram will help you to visualise the problem.

Tony walks 3 km in a westerly direction and then turns south and walks for a further 7 km. What is his bearing and how far is he from his starting point?

**EXAMPLE 2**

From an observation point 15 m from the base of the tower the angle of elevation of the top of tower is \( 32^\circ \). Find the height of the tower.

Use the tangent ratio:

\[ \tan 32^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{15} \]

where \( h \) is the height of the tower

\[ 15 \times \tan 32^\circ = h \]

\[ 15 \times 0.625 = h \]

\[ h = 9.373 \]

The height of the tower is 9.37 m.
Exercise 0.9

1. Work out the value of $x$ in each of these triangles. Where appropriate give your answers correct to 3 s.f. Diagrams are not drawn to scale.

2. Find the value of $\theta$ in each part. Where appropriate give your answers correct to 3 s.f. Diagrams are not drawn to scale.
3 Draw a diagram for each part.
Give your answers correct to three significant figures or two decimal places.

a Sonia is measuring the angle of elevation of the top of a flagpole, 9 m high.
Her eyes are 1.3 m above ground level and she is at a distance of 5 m from the flagpole.
What is the angle of elevation of the top of the flagpole from Sonia’s eye level?

b From a starting point, A, Graham cycles 3.2 km east and then 1.2 km north to reach the base of a tower.
The angle of elevation of the top of the tower from his starting point is 1.85°.
Find the height of the tower in metres.

c The vertical angle of a right circular cone is 37° and the vertical height of the cone is 9.8 cm.
What is the base radius of the cone?

d The longest diagonal in a cuboid measures 17.8 cm and makes an angle of 21° with the bottom face of the cuboid.
Find the height of the cuboid.
If one of the base edges of the cuboid measures 3.45 cm
find the length of the other base edge.

e The radius of a circle is 23 cm and an arc of the circle subtends an angle of 73° at the centre.
Find the length of the chord joining the ends of the radii.

f A simple pendulum swings between two points 35 cm apart on the same horizontal level.
The angle at the centre of oscillation is 35°.
Find the length of the string supporting the pendulum.

g Alex walks 4 km in a north-east direction to a point A.
From the same starting point as Alex, Bryony walks 7 km in a south-east direction to a point B.
Find the bearing of point A from point B.

4 Draw a diagram for each part.

a The points A (1,3) and B (3,8) are joined by a straight line.
Find the angle the extended line AB makes with the x-axis.

b A pyramid has a square base of side 9.5 cm and a vertical height of 8 cm.
Find the angle a face of the pyramid makes with the base.

c From a point A the angle of elevation of the top of a mast is 21° and from a point B the angle is 16°.
The base of the mast and the points A and B are in a straight line on horizontal ground with A and B on opposite sides of the mast.
If the height of the mast is h metres find the distance AB in terms of h.

d A cylindrical tube of base circumference 27 cm is of height 12 cm. A stick of negligible thickness just fits inside the tube at an angle.
Find the length of the stick and the angle it makes with the horizontal.

e A point A is on a bearing of 063° from a fixed point, P, and a point B is on a bearing of 133° from P.
In this position A is due north of B.
The line joining A to B is 5.35 km from P at the nearest point.
Find the distance AB.
Review O

1. Simplify these expressions expanding first where necessary.
   a) \(3b - 2c + 5c - 4b\)  
   b) \(2(a - 2b) + 3(b - 2a)\)  
   c) \(p(q - 1) + q(p - 1)\)  
   d) \(a + b - 2(a - b)\)  
   e) \((x + 2)(x - 5)\)  
   f) \((2x - 3)(2x + 3)\)  
   g) \((y - 1)(3y - 4)\)  
   h) \(2(3y - 1)(y + 2)\)  
   i) \(x(3y - 1)x + 3)\)  
   j) \((x^2 - 1)(x + 3)\)  
   k) \((x - 1)(x + 1)(x + 2)\)  
   l) \((x + 2)(x + 3)(x + 4)\)

2. Given \(a = 4\), \(b = -5\) and \(c = 2\) find the values of these expressions.
   a) \(2b + 3a\)  
   b) \((2b)^2\)  
   c) \(\frac{c - d}{a}\)  
   d) \(\sqrt{(b^2 - a^2)}\)

3. Evaluate these formulae.
   a) \(L = \frac{T^2g}{4\pi^2}\) find \(L\) when \(T = 100\), \(g = 9.8\) and \(\pi^2 = 10\)  
   b) \(V = \frac{4}{3}\pi r^3\) find \(r\) when \(\pi = 3\) and \(V = 108\)  
   c) \(I = \frac{PTR}{100}\) find \(R\) when \(I = 50\), \(P = 1000\) and \(T = 2.5\)

4. Solve these equations.
   a) \(4(2r - 1) + 3(1 - 2r) = 0\)  
   b) \(0.7(4p - 3) = 1.4\)  
   c) \(\frac{3r}{2} + \frac{5r}{6} = 7\)  
   d) \(\frac{2}{3y + 1} = 3\)

5. Draw the graphs of these equations for \(-2 \leq x \leq 4\).
   a) \(y = -x + 4\)  
   b) \(y = x^2 - 5x + 3\)  
   c) \(y = -f(x)\) when \(f(x) = -2x + 1\)  
   d) \(y = 2f(x)\) when \(f(x) = x - 2\)

6. Solve these simultaneous equations using an algebraic method.
   a) \(2x + 3y = 7\)  
   \(5x - 3y = -14\)  
   b) \(5y + 4x = 2\)  
   \(9x + 5y = -8\)  
   c) \(3y - 4x = -8\)  
   \(y - 2x = -3\)  
   d) \(5x - 2y = 15\)  
   \(4y - x = 6\)  
   e) \(2x - 3y = 5\)  
   \(3x - 4y = -1\)  
   f) \(5x + 4y = 0\)  
   \(3x - 2y = 11\)  
   g) \(6 - 7x = 4y\)  
   \(5y = 2x + 29\)  
   h) \(10x = 3y - 2\)  
   \(7x = 2y - 1\)

7. Use a graphical method to solve these simultaneous equations.
   a) \(y = x - 2\), \(2y - x = 0\)  
   b) \(\frac{x}{2} + \frac{y}{4} = 1\), \(y = 2x - 1\)  
   c) \(2y + 3x = 1\), \(2y - 3x = -1\)  
   d) \(y + 4 = 5x, x + y = 2\)

8. a) Show the solutions of these inequalities on a number line.
   i) \(-1 < x + 1 \leq 4\)  
   ii) \(2x < 5\), \(3x - 1 \geq -7\)

   b) Show the solutions of these inequalities as a region on a graph. Shade the required regions.
   i) \(x + 3 > 4\) and \(2y - 3 \leq 5\)  
   ii) \(3y + 4 \geq 2y + 3\) and \(-1 < x < 3\)

9. a) The point \((x, y)\) has integer coordinates and lies in the region defined by \(1 < x < 3\) and \(2 < y < 4\).
   State the values of \(x\) and \(y\).

   b) Find the largest integer \(x\)-value and \(y\)-value which lies in the region defined by \(x < 2\) and \(y < 4\).

   c) The sum of two positive integers, \(a\) and \(b\), is less than 13, and the difference between the two numbers is less than 3.
   Investigate possible values of \(a\) and \(b\) if \(a > b\).

   d) Given that \(x < 2y\) shade the region on a graph to indicate where the inequality is located.

10. \(P, Q\) and \(R\) are points on a circle. \(O\) is the centre of the circle. \(RT\) is the tangent to the circle at \(R\).
   Angle \(QRT = 56\)°.
   a) Find
   i) the size of angle \(RPQ\)  
   ii) the size of angle \(ROQ\)

   A, B, C and \(D\) are points on a circle. \(AC\) is a diameter of the circle.
   Angle \(CAD = 25\)° and angle \(BCD = 132\)°.
   b) Calculate
   i) the size of angle \(RAC\)  
   ii) the size of angle \(ABD\)

11. Draw diagrams and use them to solve the following problems.
   a) Find the vertical height of an equilateral triangle whose sides are of length 7.5 cm.

   b) The diagonals of a rhombus are 12 cm and 10 cm.
   Find the size of the angles at the vertices of the rhombus.
Summary

- Algebraic expressions can be simplified when there are like terms present. Refer to 0.1
- You factorise a quadratic expression by writing it as a product of its factors. Refer to 0.1
- A formula can be evaluated by substituting the given values into it and working out the result. Refer to 0.1
- To change the subject of a formula you rewrite the equation to express the named letter in terms of the other variables. Refer to 0.2
- You can solve a linear equation by rearranging its terms. Refer to 0.2
- Take care in dealing with signs. Refer to 0.3
- The graph of a linear equation is a straight line. Refer to 0.4
- You can use Pythagoras’ theorem to find the midpoint and length of a line segment. Refer to 0.5
- Simultaneous equations can be solved by elimination, substitution, or by drawing a graph. Refer to 0.6
- Inequalities are used to define a range or a region on a graph. Refer to 0.7
- Circle theorems can be applied to find angles within circles. Refer to 0.8
- Pythagoras’ theorem can be used in a right-angled triangle if two sides are known. Refer to 0.9
- The trigonometric ratios can be used in a right-angled triangle if one side and an angle or two sides are known. Refer to 0.9