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### Introduction

This chapter covers the basics of hypothesis testing which is developed further in Chapter 21. This work will be entirely new to students as there is no reference to this type of testing at GCSE level. This work has also not been part of the first statistics modules of pre-2017 specifications so many teachers will not have taught this before.

The basis of hypothesis testing is to test a whether a claim or hypothesis should be accepted or rejected based on the available data. A statistical statement is made — this is known as the null hypothesis and given the notation $H_0$. A statistical test is then performed and the null hypothesis is only rejected if there is sufficient evidence to do so. If this is the case the alternative hypothesis, $H_1$, which contradicts the null hypothesis, is accepted instead.

A null hypothesis always suggests that a statistic is equal to a particular value and so includes an equals sign whereas the alternative hypothesis can say that the statistic being tested is lower, higher or simply not equal to the value stated in the null hypothesis.

All hypothesis tests have a significance level. The lower it is, the more evidence you need to reject the null hypothesis, and the lower is the chance that a wrong decision will be made. At this stage students are given the critical region and values. They do not have to calculate them.

It is very important that the early basic ideas in the chapter are understood by all students otherwise the rest of the work will be built on very shaky foundations. It is therefore worth taking the early stages slowly to ensure full understanding.

Chapter 11 has two sections.
Chapter content

11.1 Formulating a test

The null hypothesis is a statement with an equals sign in it. Here, $H_0$ states that $p = 0.85$ to represent the 85% of clients who eat less than 42 g of butter per week.

The alternative hypothesis contradicts the null hypothesis, but changes depending on the belief of which way the null hypothesis is wrong.

- If you feel that 85% is an overestimate, then $H_1$ states that $p < 0.85$
- On the other hand, if you think 85% is an underestimate, then $H_1$ states that $p > 0.85$

Both of these will be tested with a one-tailed test as you just want to look either above or below the given value.

- If you just want to say that 85% is wrong, then $H_1$ states that $p \neq 0.85$

This will be tested using a two-tailed test as you are looking both below and above the given value.

Once the test has been set up, the next stage is to get the critical values. These are the values that lie on the edge of the critical region. You can then look to see if you should accept or reject the null hypothesis.

This is demonstrated in Example 3. Note that in this section the critical values are given and students are not required to calculate them.

This is an example of a one-tailed test because the value is thought to be too high. Thus the alternative hypothesis is that $p < 0.7$

Students are told that the critical value at a 10% significance level is 14.

This means that the critical region is $X \leq 14$ and the acceptance region is $X \geq 15$. That is, if 14 or fewer candidates in the sample pass their driving test then the null hypothesis should be rejected whereas if 15 or more candidates in the sample pass, the null hypothesis should be accepted.

11.2 The critical region

This section requires students to calculate the critical values themselves and so identify the critical region. The examples are based on the binomial distribution covered in Chapter 10. If these two chapters are not taught consecutively, you may wish to revise the binomial distribution first.

In these problems students will be presented with a binomial distribution with a number of trials given. The null hypothesis will be a particular value of $p$ to go with the distribution. The alternative hypothesis will then
be that \( p \) is either greater than, less than or not equal to this value. Given a significance level for the test, usually 1%, 5% or 10%, students then calculate a value of \( X \) which gives this percentage using the binomial formula. Since it is very unlikely that this will produce a probability of exactly 5% = 0.05, students will need to choose a value that makes the value no more than 5%, that is, as close to 5% as possible without exceeding it. In some questions, students are just asked to work out a particular value and check whether this would be in the critical region or not.

An alternative to finding the critical values is to use the \( p \)-value of the result obtained. Example 2 in this section illustrates this method.

The baker’s claim that 80% of loaves sold are white gives the null hypothesis of \( p = 0.8 \)

Because the assistant just says that it is inaccurate but does not specify too low or too high, the alternative hypothesis becomes \( p \neq 0.8 \)

The significance level required is 10% so the probability from the binomial distribution is required to be as close to 0.1 as possible whilst not exceeding it. Since it is a two-tailed test, this value is halved as you would want 5% at either side of the distribution. The assistant finds that 26 out of the 28 loaves are white so find \( P(X \geq 26) \) from the binomial distribution. \( X \sim B(28, 0.8) \). As this comes to 0.061, this is more than 5%. This means that there is not sufficient evidence to reject the null hypothesis and so it should be accepted.

**Investigation—Biased dice?**

Ask each student or group of students to create a biased dice. This can be done easily using the net of a dice with a piece of Blu-Tak taped onto an internal face of the dice. The dice will then be biased to the number opposite the face with the Blu-Tak on it.

Dice should then be swapped between students or groups.

**Question** How do you decide if a dice is biased?

It is likely that students will suggest rolling the dice many times and looking at the relative frequencies of the faces 1 to 6. Given such data, take the opportunity to quiz the students on how they would decide if the dice is biased—press them to be quantitative.

This should provide the opportunity to cover the material in the chapter naturally.

- Section 11.1 explains how to formulate null and alternative hypotheses to test against. The null hypothesis is that the dice is not biased and so the probability of getting any number is \( H_0: p = \frac{1}{6} \)

  The alternative hypothesis will be that either \( H_1: p > \frac{1}{6} \) or \( p < \frac{1}{6} \) depending on what is being tested.

- Given a critical region, section 11.1 shows students how to use their results to accept or reject the null hypothesis. This can be repeated with different significance levels to bring out the idea that the lower the significance level, the more evidence is needed to reject \( H_0 \) and the lower is the probability of making a wrong decision assuming \( H_0 \) is correct.

- Students can then investigate how to find critical values for themselves, using section 11.2 for guidance. Students should use their calculators to find the cumulative probabilities for a given binomial distribution.

- Section 11.2 also explains how to use the \( p \)-value to test a hypothesis in the absence of knowing the critical values.
11.1 Formulating a test

Simplification

Example 1

The proportion of ten pence pieces in a large collection of coins is thought to be 24%.
An independent random sample of size 12 is taken from the collection and the number, \( X \), of ten pence pieces is noted.

a. Write down the distribution of \( X \) if the proportion of ten pence pieces is \( p \).
b. Give the null and alternative hypotheses for a test of this belief against possibility that the proportion may be less than 24%.
c. i. Describe in words the critical region.
   ii. If \( X \) lies in the critical region, then state your conclusion.

\[
\begin{align*}
\text{a.} & \quad X \sim B(12, p) \\
\text{b.} & \quad H_0: p = 0.24 \\
& \quad H_1: p < 0.24 \\
\text{c.} & \quad \text{i. The critical region is the left hand tail of the distribution—low values of } X \\
& \quad \text{ii. Reject } H_0 \\
& \quad \text{The evidence suggests that the fraction of ten pence pieces is less than 24%}
\end{align*}
\]

There are twelve trials, each of which is independent of one another and otherwise identical. There are only two outcomes: yes it is a ten pence piece, with probability \( p \), or no it is not a ten pence piece. These are the conditions for a binomial distribution.

The null hypothesis always contains an equals sign. You are testing against the probability being less than 24% so the alternative hypothesis is one-tailed and contains a less than sign.

The critical region should match with the alternative hypothesis.

A value in the critical region \( \Rightarrow \) the null hypothesis is rejected.

Example 2

A manufacturer states that 40% of the chocolates in a large tin are dark.
A customer thinks the fraction is different. To test this they take 20 chocolates from a tin and find that 12 are dark.

a. State the hypotheses being tested.
b. If \( N \sim B(20, 0.4) \) then \( P(N \geq 12) = 0.0565 \)
Assuming a 10% significance level, state with reasons the conclusion of the test.

\[
\begin{align*}
\text{a.} & \quad \text{Let } f = \text{the fraction of dark chocolates} \\
& \quad H_0: f = 0.4 \\
& \quad H_1: f \neq 0.4 \\
\text{b.} & \quad \text{The critical region is two-tailed with each tail containing less than or equal to 5% probability. } 0.0565 > 0.05 \\
& \quad \text{The null hypothesis is accepted.}
\end{align*}
\]

Since the fraction is thought to be different the alternative hypothesis contains a not equals sign.

Not equals \( \Rightarrow \) a two-tailed test, with the probability of rejection shared equally.
1 A variety of rose produces red or white flowers. A supermarket buyer wishes to buy roses where the proportion of white flowers is at least 75%. She takes a sample of 30 roses to test if the proportion of plants with white flowers is sufficiently high.
   a If \( p \) is the population proportion of plants with white flowers, give the distribution of \( X \), a random variable for the number of plants in the sample with white flowers.
   b State the null and alternative hypothesis for this test.
   c If the critical region is in the left or right tail of the distribution in \( H_a \) and, if the observed value of \( X \) falls in the critical region, state the final conclusion of the test.

2 A dice is thrown 20 times and the number of sixes recorded. A test to whether or not the dice is fair to be performed. State fully the null and alternative hypotheses for the test for these questions.
   a Is the dice biased?
   b Is the dice biased towards sixes?
      In each case, state whether it is low, high or both low and high values of the total number of sixes which lead to rejection of the null hypothesis.

3 21.2% of Germans and 18.8% of Austrians are over 65 years old. Explain why it is not appropriate to perform a hypothesis test to investigate whether these proportions are significantly different.

4 In a large survey carried out five years ago, 6 out of 10 adults agreed that spending on all food and drink had decreased over the last 15 years. This year, in an independent random sample of 30 adults, 23 agreed with this statement.
   Let \( N \) be the number of adults in the sample who agreed with the statement.
   a State the null and alternative hypotheses for a test that the proportion agreeing with the statement had changed.
   b Given that, under the null hypothesis, \( P(N \geq 23) = 0.044 \) (3 dp), is there evidence at a 5% significance level that the proportion of adults holding this view has changed? Give a reason for your answer.

5 You wish to investigate whether a coin is biased towards heads. You toss the coin four times and note the number, \( X \), of heads showing.
   a Write down the null and alternative hypotheses to test this possibility.
   b The table shows the probabilities of all outcomes of the experiment.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>0.0625</td>
<td>0.25</td>
<td>0.375</td>
<td>0.25</td>
<td>0.0625</td>
</tr>
</tbody>
</table>

   Explain why this test could not lead to a rejection of the null hypothesis at a significance level of 5%. How could the test be changed to make this possible?

6 It is estimated that 40% of adults in the UK agree that the amount they spend on olive oil each week has increased over the last 4 years. A sample of 30 adults were questioned and 15 reported an increase in this spending.
   a State a condition on the method of choosing the sample so that a binomial probability model can be used to test the estimate.
   b If the condition in part a is met, write down the null and alternative hypotheses.
      Explain your choice of alternative hypothesis.

7 In a large city, the proportion of individuals with a fat intake more than the recommended daily allowance was found to be 28%. In a sample of 50 individuals taken 5 years later, 20 had a fat intake more than recommended. A statistical test with a significance level of 5% is to be carried out to determine whether the proportion with a fat intake higher than recommended has increased over this period.
   a State, with reasons, a condition for the sample to be suitable for use in the test.
   b Let \( X \) = the number of individuals in the sample with a fat intake higher than the recommended.
      State \( H_0 \) and \( H_1 \) for this test and, given that \( H_0 \) is true, give the distribution of \( X \).
   c If \( P(X \geq 20) = 0.045 \), what is the conclusion of the hypothesis test?
Extension

Example 1

In 2010, 7 out of every 10 residents of a London borough purchased more than 90 g of confectionary per week. 5 years later a survey of \( n \) people was taken and \( X \), the number of people in the sample with this level of spending on confectionary was found to be 26.

a. Using the table of probabilities below, decide whether there is evidence at a significance level of 5% that the proportion of people with this spending has

i. Increased if \( n = 30 \)

ii. Increased if \( n = 31 \)

iii. Changed if \( n = 30 \)

In each case you should write down your null and alternative hypotheses.

n | P(X ≥ 26)  
---|---------
30 | 0.0302  
31 | 0.0627

b. Comment on your answers to parts i and ii above.

The table shows probabilities associated with a binomial distribution with \( p = 0.7 \) for two values of sample size \( n \).

a. \( X \sim B(30, p) \)

i. \( H_0: p = 0.7 \) and \( H_1: p > 0.7 \)

Under \( H_0 \), \( P(X \geq 26) = 0.0302 \)

0.0302 < 0.05 so reject the null hypothesis that \( p = 0.7 \)

ii. \( H_0: p = 0.7 \) and \( H_1: p > 0.7 \)

Under \( H_0 \), \( P(X \geq 26) = 0.0627 \)

0.0627 < 0.05 so reject the null hypothesis that \( p = 0.7 \)

b. Conclusions are different because a larger sample will increase probabilities of large \( x \)-values and therefore in part ii taking the observed value out of the critical region.

Compare with 0.05 because this is a 5% one-tailed test. A one-tailed test because \( H_1: p > 0.7 \)

Compare with 0.025 because this is a 5% two-tailed test.

Example 2

In a test of the hypotheses \( H_0: p = 0.4 \) and \( H_1: p \neq 0.4 \) at a significance level of 10% the critical values are 5 and 15.

a. Find an inequality for \( P(4 < X < 11) \)

b. If \( a \) and \( b \), \( a < b \), are the critical values for the same test with a significance level of 20% find inequalities for \( a \) and \( b \)

a. A two-tailed test

Each tail \( 10 \div 2 = 5\% \)

\( P(X \leq 4) \leq 5\% \)

\( P(X \geq 11) \leq 5\% \)

\( P(4 < X < 11) = 1 - P(X \leq 4) - P(X \geq 11) \)

\( \geq 1 - 0.05 - 0.05 \)

\( \geq 0.90 \)

b. \( 4 \leq a < b \leq 11 \)

Use a diagram to clarify your thoughts.

The 10% critical regions will be contained within the 20% critical regions.
Extension questions 11.1

1 In hypothesis testing, it is usual to either 'reject the null hypothesis' or state that there is 'no reason to reject the null hypothesis.'
   Why is this language used rather than the more obvious 'reject' or 'accept'?

2 a State the conditions under which the binomial distribution provides a good model for a statistical experiment.
   In 2013 in the North West of England, the median amount of boiled sweets purchased was 35 g per person per day. To test the hypothesis that more boiled sweets are bought in the North East of England, researchers sampled 25 adults from the North East and found that 18 purchased more than 35 g.
   b Why is it necessary to know the median amount spent on food and drink rather than either the mean or mode?
   c Let X be the number in the sample purchasing more than 35 g. Give the distribution of X and state the null and alternative hypotheses for the required test.
   d Given that under the null hypothesis $P(X < 18) = 0.978$, write down the result of the test and the probability that the conclusion is incorrect given that the null hypothesis is true.

3 In 2010, 12.2% of the residents in a town in England purchased more than 280 ml of mineral or spring water per week. You wish to investigate a possible change in this proportion. In 2015, a sample of 50 individuals in England included 9 individuals with this level of spending. Let X be a random variable for the number in the sample who buy more than 280 ml per week.
   a State the null and alternative hypotheses for a suitable hypothesis test with a significance level of 5% if you are testing for
      i A change in the proportion,
      ii An increase in the proportion.
   b Explain why it makes no sense to test whether there is a decrease in this proportion.
   c In both of the tests in part a, a large value of X will lead to the null hypothesis being rejected. Explain why the lower limit for rejection is higher for the test in part a.

4 In a test of the hypotheses $H_0: p = 0.34$ and $H_1: p < 0.34$ at a significance level of 10% the critical value is 8
   a Find an inequality for $P(X \leq 8)$
   b Find an inequality for a if a is the critical value for a test at a 20% significance level.

5 In a test of the hypotheses $H_0: p = 0.43$ and $H_1: p \neq 0.43$ at a significance level of 5% the critical values are 10 and 40
   a Find inequalities for
      i $P(10 < X < 40)$
      ii $P(X < 9)$
      iii $P(9 < X < 41)$
   b If a and b, $a < b$, are the critical values for the same test with a significance level of 10% find inequalities for a and b

6 a Successive binomial probabilities can be found using the recurrence formula
   $$P(X = x + 1) = \frac{(n-x)p}{(x+1)q} P(X = x)$$
   where $X \sim B(n, p)$ and $q = 1 - p$
   You are given that if $n = 50$ and $p = 0.6$, then $P(X \leq 23) = 0.0314$ and $P(X = 23) = 0.0154$
   Using the recurrence formula, find $P(X \leq 24)$.
   b In a large survey by a national bakery, it was found that in Town A, the proportion of adults who had purchased at least 600 g of bread within the last week was 60%. To investigate national purchasing differences, an independent random sample of 50 adults from Town B was taken and 24 of these had purchased at least 600 g of bread within the last week. Use the results in part a to test at a significance level of 5% the hypothesis that Town B has a lower proportion of adults with this level of spending.
Common misconceptions / Exam tips

One or two tails

It is important that students understand the difference between a one-tailed and a two-tailed test. Using graphs may help.

In a two-tailed test, the significance value is split in half so that half is at the top of the distribution and half at the bottom. Thus, to do a 10% two-tailed test, you need 5% at the top and bottom.

Setting up the alternative hypothesis is important. If it is thought that the claim is simply wrong, then the alternative hypothesis will be \( x \neq a \) value and a two-tailed test should be used. If it is thought to be too high or too low, then the alternative hypothesis will involve an inequality and a one-tailed test should be used.

Deciding between using a one-tailed test for low values, a one-tailed test for high values or a two-tailed test, often depends on identifying the key word or phrase in the question that tells you how the probability of success may be different under the alternative hypothesis \( H_a \).

For example, if \( H_0: p = p_0 \), then these words

- *too low*, underestimate, less often, lower, reduced, worse
- *too high*, overestimate, more often, higher, increased, better
- *inaccurate*, incorrect, wrong, has changed

The actual interpretation will rely on the context of the question.

Exercise 11.1B Student Book commentary

**Question 1** To decide whether \( H_a \) includes an \(<, \neq, >\) sign look for the key word/phrase in the description of what is being tested.

a  'too low'  \( \Rightarrow H_a: p > 0.6 \)  [One-tailed, high values]

b  'too high'  \( \Rightarrow H_a: p < 0.6 \)  [One-tailed, low values]

c  'inaccurate'  \( \Rightarrow H_a: p \neq 0.6 \)  [Two-tailed, low and high values]

**Question 2** The key to this question is in understanding what is the critical region; if students struggle, then ask them to discuss it with a partner.

**Question 3** Parts a and b are straightforward. In part b, ensure that students do not simply write

'39 < 41 \Rightarrow accept \( H_0 \)’ but instead write a full sentence explaining their conclusion in the context of the question. Students who struggle with part c should be directed to example 4 in the Student Book.

**Question 4** Part d is a new question type. The key idea is that the size of the critical region decreases as the significance level decreases and vice versa. In the question the critical region gets smaller so you are less likely to reject \( H_0 \) by mistake and so the significance level must be less.

Encourage students to discuss the reasoning behind their answers.

**Question 5** This question extends the ideas in question 4d from a one-tailed test to a two-tailed test however the basic reasoning remains the same.

Students who finish early could be asked to write their own ‘inequality’ questions and swap them with a partner.
11.2 The critical region

Simplification

Example 1

For each hypothesis

i. Calculate the critical region(s) for a significance level of 10%
   ii. State the conclusion if \( X = 4 \)

a. \( \chi \sim \text{B}(20, p) \), \( H_0: p = 0.4 \) and \( H_1: p < 0.4 \)

b. \( \chi \sim \text{B}(12, p) \), \( H_0: p = 0.2 \) and \( H_1: p > 0.2 \)

c. \( \chi \sim \text{B}(30, p) \), \( H_0: p = 0.45 \) and \( H_1: p \neq 0.45 \)

\( \chi \):

- Under \( H_0, \chi \sim \text{B}(20, 0.4) \)
  - One-tailed test, low values of \( \chi \)
  - \( P(\chi \leq 4) = 5.09\% \) and \( P(\chi \leq 5) = 12.5\% \)
  - Critical region \( \chi \leq 4 \)

- Under \( H_0, \chi \sim \text{B}(12, 0.2) \)
  - One-tailed test, high values of \( \chi \)
  - \( P(\chi \geq 4) = 20.5\% \) and \( P(\chi \geq 5) = 7.25\% \)
  - Critical region \( \chi \geq 5 \)

- Under \( H_0, \chi \sim \text{B}(30, 0.45) \)
  - Two-tailed test, low and high values of \( \chi \)
  - Probability in each tail \( \leq \frac{10}{2} = 5\% \)
  - \( P(\chi \leq 8) = 3.12\% \) and \( P(\chi \leq 9) = 6.94\% \)
  - \( P(\chi \geq 18) = 7.13\% \) and \( P(\chi \geq 19) = 3.34\% \)
  - Critical region \( \chi \leq 8 \) and \( \chi \geq 19 \)

- 4 is in the critical region \( \Rightarrow \) reject \( H_0 \)

Example 2

A survey of the public asked respondents whether or not they purchased white bread. Five years ago, 76% of people said they did. This year, in an independent random sample of 20 people, 12 said they did. Is there evidence that the proportion of people eating white bread has decreased? You should use a 10% level of significance and give the exact size of the critical region.

- \( H_0: p = 0.76 \) and \( H_1: p < 0.76 \)
- \( \chi \sim \text{B}(20, 0.76) \)
- \( P(\chi \leq 12) = 8.34\% \) and \( P(\chi \leq 13) = 18.3\% \)
- Critical region \( \chi \leq 12 \)

- 12 is in the critical region \( \Rightarrow \) reject \( H_0 \)

- OR
  - \( p\)-value = \( P(\chi \leq 12) = 8.34\% \)

- \( p\)-value \( < 10\% \Rightarrow \) reject \( H_0 \)

There is evidence that the proportion of people purchasing white bread has decreased.
Simplification questions 11.2

1. For each hypothesis
   i. Find the critical region(s) for a significance level of 10%.
   ii. State the conclusion for the given value of $X$
      a. $X \sim B(20, p)$, $H_0: p = 0.3$, $H_1: p < 0.3$ and $X = 3$
      b. $X \sim B(10, p)$, $H_0: p = 0.5$, $H_1: p > 0.5$ and $X = 2$
      c. $X \sim B(40, p)$, $H_0: p = 0.55$, $H_1: p \neq 0.45$ and $X = 16$

2. The random variable $X \sim B(20, p)$. Annie claims that $p = 0.5$ but Barry says $p$ is higher.
   A test gives $X = 14$
   a. State the null and alternative hypotheses for the test.
   b. For a significance level of 10%, state the critical region(s).
   c. Given the result, should you accept Annie’s claim?

3. A random variable $X$ has a binomial distribution, parameters $n = 8$ and $p$, a constant. The value of $p$ was known to be 0.7 but is now believed to have changed.
   a. Write down the null and alternative hypotheses in a test of this belief.
   b. Using a significance level of 5%, what values of $X$ would suggest that the belief is incorrect?

4. A fortune teller claims that in more than 70% of cases, he can predict the marital status of his client. He is tested with a random selection of 8 clients and successfully identifies the marital status of all of them.
   Using the probability distribution in part a, perform a hypothesis test of his claim using a significance level of 10%.
   You should state clearly the null and alternative hypotheses.

5. In a large investigation carried out in 2013, 6 out of every 10 people questioned agreed that their monthly spending on food had increased over the last year. This year, in an independent random sample of 15 people, 12 agreed with this proposition. A test was carried out to determine whether the proportion of people agreeing with the proposition had increased within the whole population.
   a. Write down the null and alternative hypotheses for the test.
   b. Perform the test at a significance level of 10%, stating clearly the conclusion of the test.

6. In 2007 the median household expenditure on all food and drink among low income households was £34.57 per person per week. In order to test whether this had increased by 2013, 20 similar households were asked about their food and drink expenditure. Fourteen households agreed that spending per person was above £34.57. Test at a significance level of 5% whether this is evidence of an increase among all low income families.
   a. Copy and complete the table for binomial probabilities with $n = 8$, $p = 0.44$
      Give your answers to 4 dp.
   b. It is estimated that 56% of adults living in rural areas of the UK purchase more than 2 litres of cream and milk per week, with urban dwellers purchasing less than this. In a survey of 8 adults from urban areas, 6 said they purchased less than 2 litres per week.
      Perform a test with a significance level of 5% to investigate whether urban dwellers spend less on cream and milk than those in rural areas. You should state clearly your null and alternative hypotheses.
Example 1

A newspaper claimed that 1 in 4 adults in the UK are obese. A researcher thinks that this is an underestimate. He decides to carry out a hypothesis test with a 5% significance level using a sample of 40 adults from town A.

a State the null and alternative hypotheses.

b In the sample $n$ of the adults were obese and he accepts $H_0$ for town A.

He then repeats the survey in town B and finds $n + 1$ adults are obese and rejects $H_0$ for town B

What is the value of $n$?

\[
\begin{align*}
\text{Let } p &= \text{ the fraction of obese adults} \\
H_0: p &= 0.25 \quad H_1: p > 0.25 \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
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\end{align*}
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Example 2

It is estimated that the proportion of adults in Yorkshire who purchase margarine is 0.4. To test this estimate, a sample of $n$ adults from Yorkshire is taken and, $X$, the number who support this belief is noted. The set of $x$-values which leads to a rejection of the estimate at a significance level of 5% is $x \leq 6, x \geq 18$

a Show that $n = 30$

b Explain why it may be useful to consider a test with a smaller significance level.

c Find the critical region for the same test with a significance level of 2%.

\[
\begin{align*}
a \quad X &\sim B(n, p) \text{ where } p \text{ is the proportion of adults out of a total of } n \text{ who support the belief.} \\
H_0: p = 0.4 \text{ and } H_1: p \neq 0.4 \\
\text{Under } H_0, \text{ if } n = 30, X \sim B(30, 0.4) \\
\text{then, } P(X \leq 6) = 0.017 \ldots \text{ and } P(X \leq 7) = 0.043\ldots \\
\text{and } P(X \geq 18) = 0.021 \text{ and } P(X \geq 19) = 0.048\ldots \\
\text{Therefore, critical region is } x \leq 6, x \geq 18 \text{ as stated.} \\
\text{Therefore } n = 30 \\
b \quad \text{The significance level of a test is equal to the probability that the null hypothesis is rejected when it is true.} \\
\text{Reducing this probability may be of particular importance if the consequences of making this mistake are serious.} \\
c \quad P(X \leq 5) = 0.005 \ldots \text{ and } P(X \leq 6) = 0.017\ldots \\
\text{and } P(X \geq 19) = 0.005 \ldots \text{ and } P(X \geq 18) = 0.021\ldots \\
\text{Critical region: } x \leq 5, x \geq 19 
\end{align*}
\]
Extension questions 11.2

1 a A hypothesis test is carried out for the null hypothesis that a population fraction equals \( p_o \), against the alternative that it is less than \( p_o \). Explain why a one-tailed test, with the critical region on the left of the distribution defined in the null hypothesis, is used and not a two-tailed test.

b The proportion of men aged between 45 and 64 in England who buy more brown or wholemeal bread than white bread is 25.2%. A sample of adults in the North East of England was taken to test at a significance level of 5% whether they were typical of the English population.

i Write down the null and alternative hypotheses for this test.

ii Find the maximum sample size if there were 12 men who bought mostly brown bread in the sample and the null hypothesis was rejected.

2 In a town in the North East, the proportion of adults who purchase yoghurt is 0.5. It is believed that this proportion is lower in a town in the North West. A sample of 20 adults from the North West is taken and included 7 who don’t purchase yoghurt. A researcher wishes to perform a hypothesis test of this belief at a significance level of 5%.

a State the null and alternative hypotheses for this test.

b Find the probability that, under the null hypothesis, fewer than or equal to 7 adults do not purchase yoghurt.

c State the conclusion about the null hypothesis to be drawn from part b.

3 A firm wishes to sell a large batch of electrical components. A buyer decides to accept the batch if a random sample of 30 components contains no more than one defective component.

a If the probability of the batch being accepted has to be at least 95%, show that the maximum probability that one component is defective is 0.011 (3 dp).

b To test whether the probability of any item being defective is 0.011, 300 components are tested and 5 are found to be defective. Perform this test against the alternative hypothesis that the fraction of defective components is greater than 0.011.

4 A five-sided spinner, labelled 1 – 5, is spun \( n \) times and \( X \), the number of ones and twos obtained, is recorded. The spinner is tested to see if it is fair at the 20\% significance level. The critical region is \( X \leq 2 \) and \( X \geq 11 \).

a What is the value of \( n \)?

b What is the critical region for a test at the 5\% significance level?

5 A dice is rolled \( n \) times and \( X \), the number of fives and sixes obtained, is recorded. The dice is tested to see if it is fair at the 10\% significance level. The critical region is \( X \leq 4 \), \( X \geq 13 \).

Find the critical region for a test at the 5\% significance level.

6 The random variable \( X \sim B(n, p) \). A test of the hypotheses \( H_0: p = 0.3 \) and \( H_1: p > 0.3 \) is carried out for a significance level of 10\%. The result \( X = 11 \) gives a \( p \)-value that only just falls in the rejection region.

If the test had a significance level of 20\%, would \( X = 10 \) lead to \( H_0 \) being rejected?

7 In 2012, in a town in South East England, 73\% of adults purchased more than 28 g of butter per week. Later surveys suggest that this proportion has decreased. To test this hypothesis, researchers took a sample of adults from the town.

a If \( n \) is the sample size and \( X \) is the number in the sample who purchased more than 28 g of butter, write down \( H_0 \) and \( H_1 \) for this test.

b If \( n = 8 \), find the values contained in the critical region.

c Given that the critical region is \( x \leq 6 \), find the value of \( n \).

Hypothesis testing 1 The critical region

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Common misconceptions / Exam tips

Calculating critical values

In the exam AQA will not provide tables of cumulative binomial distributions. It is therefore essential that students learn how to find critical values using their own calculators. Students should be made aware that for the binomial distribution, $X \sim B(n, p)$, if a calculator can find inverse values then is likely to give the value of $x$ such that $P(X \leq x)$ is closest to the required significance level. This may mean that $P(X \leq x)$ is slightly larger than the significance level and that the critical value is $x - 1$. The correct value can be found by inspecting the value of $P(X \leq x)$ and $P(X \leq x - 1)$.

For example, suppose $X \sim B(20, 0.3)$, and a 10% significance level is required for a low-values, one-tailed test. A calculator may give $x = 3$ as $P(X \leq 3) = 10.7\ldots\%$, whereas the correct critical value is $x = 2$ for which $P(X \leq 2) = 3.54\ldots\%$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P(X \leq n)$</th>
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<tbody>
<tr>
<td>0</td>
<td>0.000798</td>
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<tr>
<td>1</td>
<td>0.007637</td>
</tr>
<tr>
<td>2</td>
<td>0.035483</td>
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<tr>
<td>3</td>
<td>0.107087</td>
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<tr>
<td>4</td>
<td>0.237508</td>
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<tr>
<td>5</td>
<td>0.416371</td>
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</tbody>
</table>

There are websites that will help you calculate these values such as www.stattrek.com

Exercise 11.2B Student Book commentary

**Question 1** In part a, the word ‘overestimate’ signals that the alternative hypothesis is $H_1$: $p < 0.8$ and a one-tailed test is required. The probability distribution is $X \sim B(40, 0.8)$.

Students should use their calculators to find the critical region, that is the smallest value of $x$ such that $P(X \leq x)$ is less than 5%; $P(X \leq 28) = 0.0875 > 5\%$ and $P(X \leq 27) = 0.0432 < 5\%$.

Part b, it is not sufficient to state ‘31 > 27, accept $H_o$’; a full sentence is required.

**Question 2** In part a, the word ‘inaccurate’ signals that alternative hypothesis is $H_1$: $p \neq 0.6$ and a two-tailed test is required. The probability distribution is $X \sim B(75, 0.6)$. In a two-tailed test the probability of rejecting $H_o$, the significance level, is split equally between the tails. $P(X \leq 35) = 0.0133 > 1\%$ but $P(X \leq 34) = 0.0072 < 1\%$ and $P(X \leq 55) = 0.0112 > 1\%$ but $P(X \leq 56) = 0.0056 < 1\%$.

**Question 3** The word ‘changed’ signals that alternative hypothesis is $H_1$: $p \neq 0.15$ and a two-tailed test is required. The probability distribution is $X \sim B(60, 0.15)$. The actual test is based on using a p-value.

$P(X \leq 4) = 0.0424 < 5\% \Rightarrow$ reject $H_o$ and $P(X \leq 5) = 0.0968 > 5\% \Rightarrow$ accept $H_o$

**Question 4** This is a one-tailed test with $X \sim B(4, 0.5)$ and $X \sim B(24, 0.5)$ with $H_o$: $p = 0.5$ and $H_1$: $p > 0.5$. In part a i, you need $P(X = 4) = 0.109 > 0.1 \Rightarrow$ accept $H_o$. In part a ii, you need $P(X \geq 17) = 0.0319 < 0.1 \Rightarrow$ reject $H_o$ (Note 16 heads would not be more than 2 × tails).

Strictly the result might be 17, 18,..., 24 heads but once rejected for 17 it will be rejected for any higher number as the p-value decreases.

**Question 5** In part a, the sum of probabilities $= 1 = 15k^2 + 5k + 0.35 \Rightarrow k = 0.1$ or −0.5

Using the positive value of $k$, the lower bound is $P(X = 0$ or 6) = 0.05 + 0.05 = 10\%.

[An (unattained) upper bound would be $P(X = 0$, 1, 5 or 6) = 0.05 + 0.2 + 0.2 + 0.05 = 50\%.

**Question 6** In part b, for the distribution $X \sim B(40, 0.1)$ you need to find $x$ such that $P(X \leq x) \leq 0.1$ and $P(X \leq x + 1) \geq 0.1$. Testing values give $x = 1$

**Question 7** This is a new type of ‘inverse’ problem. You need to find $n$ in the distribution $X \sim B(n, 0.5)$ given that $P(X \leq 12) \leq 0.01$ but $P(X \leq 13) > 0.01$ and $P(X \geq 28) \leq 0.01$ but $P(X \geq 27) > 0.01$. You should use trial and improvement to find the value of $n$.