A Caribbean Examinations Council® Study Guide

Developed exclusively with the Caribbean Examinations Council®, this study guide will provide candidates in and out of school with additional support to maximise their performance in CAPE® Pure Mathematics.

Written by an experienced team comprising teachers and experts in the CAPE® Pure Mathematics syllabus and examination, this study guide covers the elements of the syllabus that you must know in an easy-to-use double-page format. Each topic begins with the key learning outcomes from the syllabus and contains a range of features designed to enhance your study of the subject, such as:

- **Examination tips** with essential advice on succeeding in your assessments
- **Did You Know?** boxes to expand your knowledge and encourage further study
- **The key terms** you need to know
- **Practice questions** to build confidence ahead of your examinations.

This comprehensive self-study package includes a fully interactive CD, incorporating multiple-choice questions and sample examination answers with accompanying examiner feedback, to build your skills and confidence as you prepare for the CAPE® Pure Mathematics examination.

The Caribbean Examinations Council (CXC®) has worked exclusively with Nelson Thornes to produce a series of Study Guides across a wide range of subjects at CCSLC®, CSEC® and CAPE®. Developed by expert teachers and resource persons, these Study Guides have been designed to help students reach their full potential as they study their CXC® programme.
## Contents

<table>
<thead>
<tr>
<th>Introduction</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1 Basic algebra and functions</td>
<td></td>
</tr>
<tr>
<td>1.1 Terminology and principles</td>
<td>6</td>
</tr>
<tr>
<td>1.2 Binary operations</td>
<td>8</td>
</tr>
<tr>
<td>1.3 Surds</td>
<td>12</td>
</tr>
<tr>
<td>1.4 Logic and truth tables</td>
<td>14</td>
</tr>
<tr>
<td>1.5 Direct proof</td>
<td>18</td>
</tr>
<tr>
<td>1.6 Proof by induction</td>
<td>20</td>
</tr>
<tr>
<td>1.7 Remainder theorem and factor theorem</td>
<td>22</td>
</tr>
<tr>
<td>1.8 Factors of $a^n - b^n$, $n \leq 6$</td>
<td>24</td>
</tr>
<tr>
<td>1.9 Quadratic and cubic equations</td>
<td>26</td>
</tr>
<tr>
<td>1.10 Curve sketching</td>
<td>30</td>
</tr>
<tr>
<td>1.11 Transformation of curves</td>
<td>32</td>
</tr>
<tr>
<td>1.12 Rational expressions</td>
<td>36</td>
</tr>
<tr>
<td>1.13 Inequalities – quadratic and rational expressions</td>
<td>38</td>
</tr>
<tr>
<td>1.14 Intersection of curves and lines</td>
<td>42</td>
</tr>
<tr>
<td>1.15 Functions</td>
<td>44</td>
</tr>
<tr>
<td>1.16 Types of function</td>
<td>48</td>
</tr>
<tr>
<td>1.17 Inverse function</td>
<td>50</td>
</tr>
<tr>
<td>1.18 Logarithms</td>
<td>52</td>
</tr>
<tr>
<td>1.19 Exponential and logarithmic equations</td>
<td>54</td>
</tr>
<tr>
<td>1.20 Exponential and logarithmic functions</td>
<td>56</td>
</tr>
<tr>
<td>1.21 Modulus functions</td>
<td>58</td>
</tr>
<tr>
<td>1.22 Modulus equations and inequalities</td>
<td>60</td>
</tr>
<tr>
<td>Section 1 Practice questions</td>
<td>62</td>
</tr>
<tr>
<td>Section 2 Trigonometry, geometry and vectors</td>
<td></td>
</tr>
<tr>
<td>2.1 Sine, cosine and tangent functions</td>
<td>64</td>
</tr>
<tr>
<td>2.2 Reciprocal trig functions</td>
<td>68</td>
</tr>
<tr>
<td>2.3 Pythagorean identities</td>
<td>70</td>
</tr>
<tr>
<td>2.4 Compound angle formulae</td>
<td>72</td>
</tr>
<tr>
<td>2.5 Double angle identities</td>
<td>76</td>
</tr>
<tr>
<td>2.6 Factor formulae</td>
<td>78</td>
</tr>
<tr>
<td>2.7 The expression $a \cos \theta + b \sin \theta$</td>
<td>80</td>
</tr>
<tr>
<td>2.8 Trigonometric identities and equations</td>
<td>82</td>
</tr>
<tr>
<td>2.9 General solution of trig equations</td>
<td>86</td>
</tr>
<tr>
<td>2.10 Coordinate geometry and straight lines</td>
<td>90</td>
</tr>
<tr>
<td>2.11 Loci and the equation of a circle</td>
<td>92</td>
</tr>
<tr>
<td>2.12 Equations of tangents and normals to circles</td>
<td>94</td>
</tr>
</tbody>
</table>
## Contents

### Section 2  Parametric equations
- 2.13  Parametric equations 96
- 2.14  Conic sections 98
- 2.15  The parabola 100
- 2.16  The ellipse 102
- 2.17  Coordinates in 3-D and vectors 104
- 2.18  Unit vectors and problems 108
- 2.19  Scalar product 110
- 2.20  Equations of a line 112
- 2.21  Pairs of lines 116
- 2.22  Planes 118
- Section 2 Practice questions 122

### Section 3  Calculus 1
- 3.10  Rates of change 144
- 3.11  Increasing and decreasing functions 146
- 3.12  Stationary values 148
- 3.13  Determining the nature of stationary points 150
- 3.14  Curve sketching 154
- 3.15  Tangents and normals 158
- 3.16  Integration 160
- 3.17  Integration of sums and differences of functions 162
- 3.18  Integration using substitution 164
- 3.19  Calculus and the area under a curve 166
- 3.20  Definite integration 168
- 3.21  Area under a curve 170
- 3.22  Area below the x-axis and area between two curves 172
- 3.23  Volumes of revolution 174
- 3.24  More volumes of revolution 176
- 3.25  Forming differential equations 180
- 3.26  Solving differential equations 182

### Index
- 3.27  Section 3 Practice questions 186
- 3.28  Index 188
This Study Guide has been developed exclusively with the Caribbean Examinations Council (CXC®) to be used as an additional resource by candidates, both in and out of school, following the Caribbean Advanced Proficiency Examination (CAPE®) programme.

It has been prepared by a team with expertise in the CAPE® syllabus, teaching and examination. The contents are designed to support learning by providing tools to help you achieve your best in CAPE® Pure Mathematics and the features included make it easier for you to master the key concepts and requirements of the syllabus. *Do remember to refer to your syllabus for full guidance on the course requirements and examination format!*

Inside this Study Guide is an interactive CD which includes electronic activities to assist you in developing good examination techniques:

- **On Your Marks** activities provide sample examination-style short answer and essay type questions, with example candidate answers and feedback from an examiner to show where answers could be improved. These activities will build your understanding, skill level and confidence in answering examination questions.

- **Test Yourself** activities are specifically designed to provide experience of multiple-choice examination questions and helpful feedback will refer you to sections inside the study guide so that you can revise problem areas.

- **Answers** are included on the CD for exercises and practice questions, so that you can check your own work as you proceed.

This unique combination of focused syllabus content and interactive examination practice will provide you with invaluable support to help you reach your full potential in CAPE® Pure Mathematics.
Language of mathematics

The language of mathematics is a combination of words and symbols where each symbol is a shorthand form for a word or phrase. When the words and symbols are used correctly a piece of mathematical reasoning can be read in properly constructed sentences in the same way as a piece of prose.

Many of the words used have precise mathematical definitions. For example, the word ‘bearing’ has many meanings when used in everyday language, but when used mathematically it means the direction of one point from another.

You need to be able to present your solutions using clear and correct mathematical language and symbols.

Symbols used for operators

A mathematical operator is a rule for combining or changing quantities. You are already familiar with several operators and the symbols used to describe them.

+ means ‘plus’ or ‘and’ or ‘together with’ or ‘followed by’, depending on context.

For example, $2 + 5$ means 2 plus 5 or 2 and 5,

$$a + b$$

means $a$ together with $b$ or $a$ followed by $b$.

− means ‘minus’ or ‘take away’.

For example, $2 − 5$ means 2 minus 5 or 2 take away 5.

The operators $\times$ and $\div$ also have familiar meanings.

Symbols used for comparison

The commonest symbol used for comparing two quantities is = and it means ‘is equal to’.

For example, $x = 6$ means $x$ is equal to 6.

Some other familiar symbols are $>$ which means ‘is greater than’ and $\geq$ which means ‘is greater than or is equal to’. A forward slash across a comparison symbol is used to mean ‘not’, for example, $\neq$, which means ‘is not equal to’.

Terms, expressions, equations and identities

To use comparison symbols correctly, you need to recognise the difference between terms, expressions, equations and identities.

A mathematical expression is a group of numbers and/or variables (for example, $x$) and operators. For example, $2x$, $3 − 2y$ and $\frac{5x^2}{3 − 2x}$ are expressions.

The parts of an expression separated by $+$ or $−$ are called terms. For example, $3$ and $2y$ are terms in the expression $3 − 2y$. 
An equation is a statement saying that two quantities are equal in value. For example, \(2x - 3 = 7\) is a statement that reads ‘\(2x - 3\) is equal to 7’.

This statement is true only when \(x = 5\).

Some equations are true for any value that the variable can take. For example, \(x + x = 2x\) is true for any value of \(x\). This equation is an example of an identity and we use the symbol \(\equiv\) to mean ‘is identical to’. Therefore we can write \(x + x = 2x\).

**Symbols used for linking statements**

When one statement, such as \(x^2 = 4\), is followed by another statement that is logically connected, for example \(x = \pm 2\), they should be linked by words or symbols.

Some examples of words that can be used are:

- ‘\(x^2 = 4\) therefore \(x = \pm 2\)’
- ‘\(x^2 = 4\) implies that \(x = \pm 2\)’
- ‘\(x^2 = 4\) hence \(x = \pm 2\)’

The symbols \(\therefore\) and \(\Rightarrow\) can be used to link statements, where \(\therefore\) means ‘therefore’ or ‘hence’ and the symbol \(\Rightarrow\) means ‘implies that’ or ‘gives’.

For example, \(2x - 1 = 5 \therefore x = 3\) or \(2x - 1 = 5 \Rightarrow x = 3\).

**Setting out a solution**

It is important to set out your solutions to problems using correct linking symbols or words.

It is also important that you explain the steps you take and your reasoning.

The following example shows a way of explaining the solution of the pair of simultaneous equations: \(2x + 3y = 1\) and \(3x - 4y = 10\):

\[
\begin{align*}
2x + 3y &= 1 & [1] \\
3x - 4y &= 10 & [2] \\
[1] \times 3 &\Rightarrow 6x + 9y = 3 & [3] \\
\therefore &y = -1
\end{align*}
\]

Substituting \(-1\) for \(y\) in \([1]\) gives \(2x + 3(-1) = 1\) \(\Rightarrow x = 2\).

The solution is \(x = 2\) and \(y = -1\).

Notice that the equations are numbered. This gives a way of explaining briefly what we are doing to combine them in order to eliminate one of the variables.

**Exercise 1.1**

In each question, explain the incorrect use of symbols and write down a correct solution.

1. Solve the equation \(3x - 1 = 5\)
   \[
   \begin{align*}
   3x - 1 &= 5 \\
   \Rightarrow 3x &= 6 \\
   \Rightarrow x &= 2
   \end{align*}
   \]

2. Find the value of \(A\) given that \(\sin A^\circ = 0.5\)
   \[
   \sin A^\circ = 0.5 \therefore A = 30^\circ
   \]
Binary operations

A binary operation is a rule for combining two members of a set.

For example, we can combine two members of the set of real numbers by addition, subtraction, multiplication or division. We know the rules for these operations, that is $4 + 2 = 6$, $4 - 2 = 2$, $4 \times 2 = 8$ and $4 \div 2 = 2$.

We can define other operations. For example, for $a$ and $b$, where $a$, $b$ are members of the set of real numbers, $\ast$, then $a$ and $b$ are combined to give $2a - b$.

We write this briefly as $a \ast b = 2a - b$.

Then, for example, $3 \ast 7 = 2 \times 3 - 7 = -1$.

Properties of operations

An operation, $\ast$, is commutative when $a \ast b$ gives the same result as $b \ast a$ for any two members of the set.

For example, addition on the set of real numbers is commutative because $a + b = b + a$, $a$, $b \in \mathbb{R}$.

Multiplication on the set of real numbers is also commutative because $a \times b = b \times a$, $a$, $b \in \mathbb{R}$.

However, subtraction is not commutative because, in general, $a - b \neq b - a$.

E.g. $3 - 7 \neq 7 - 3$.

Division is also not commutative because, in general, $a \div b \neq b \div a$.

E.g. $3 \div 7 \neq 7 \div 3$.

An operation is associative when any three members can be combined by operating on either the first two members or the second two members first, that is $(a \ast b) \ast c = a \ast (b \ast c)$.

For example, multiplication on the set of real numbers is associative because $(a \times b) \times c = a \times (b \times c)$, $a$, $b$, $c \in \mathbb{R}$.

E.g. $[2 \times 3] \times 4 = 2 \times [3 \times 4]$.

Addition on the set of real numbers is also associative because $(a + b) + c = a + (b + c)$, $a$, $b$, $c \in \mathbb{R}$.


However, subtraction is not associative because, in general, $(a - b) - c \neq a - (b - c)$, $a$, $b$, $c \in \mathbb{R}$.

E.g. $[2 - 3] - 4 = -1 - 4 = -5$ whereas $2 - [3 - 4] = 2 - (-1) = 3$.

Division is also not associative because, in general, $(a \div b) \div c \neq a \div (b \div c)$, $a$, $b$, $c \in \mathbb{R}$.

E.g. $[2 \div 3] \div 4 = \frac{2}{3} \div 4 = \frac{1}{6}$ whereas $2 \div [3 \div 4] = 2 \div \frac{3}{4} = \frac{8}{3}$.
An operation, $\ast$, is **distributive** over another operation, $\odot$, when for any three members of the set $a \ast \{b \odot c\} = (a \ast b) \odot (a \ast c)$

For example, multiplication is distributive over addition and subtraction on members of $\mathbb{R}$ because

$$a \times (b + c) = ab + ac$$

and

$$a \times (b - c) = ab - ac$$

but multiplication is not distributive over division because

$$a \times (b \div c) = \frac{ab}{c}$$

whereas

$$(a \times b) \div (a \times c) = \frac{b}{c}$$

so

$$a \times (b \div c) \neq (a \times b) \div (a \times c)$$

unless $a = 1$

---

**Example**

An operation $\ast$ is defined for all real numbers $x$ and $y$ as

$$x \ast y = 2x + 2y$$

Determine whether the operation $\ast$ is: (a) commutative (b) associative.

(a) $x \ast y = 2x + 2y$ and $y \ast x = 2y + 2x$

$$2x + 2y = 2y + 2x$$

because the addition of real numbers is commutative.

$\therefore$ the operation $\ast$ is commutative.

(b) Taking $x$, $y$ and $z$ as three real numbers,

$$x \ast (y \ast z) = x \ast (2y + 2z) = 2x + 4y + 4z$$

Therefore $x \ast y \ast z \neq x \ast (y \ast z)$ so the operation is not associative.

---

**Example**

(a) For two real numbers, $x$ and $y$, the operation $\ast$ is given by

$$x \ast y = x^3 + y^3$$

Determine whether the operation is associative.

(b) For two real numbers, $x$ and $y$, the operation $\odot$ is given by $x \odot y = xy$

Determine whether the operation $\odot$ is distributive over the operation $\ast$.

(a) For any three real numbers, $x$, $y$ and $z$, 

$$x \ast (y \ast z) = (x^3 + y^3) \ast z = (x^3 + y^3)^2 + z^2$$

$$x \ast (y \ast z) = x^3 \ast (x^2 + z^2) = x^2 + (x^2 + z^2)^2$$

$\therefore$ ($x \ast y) \ast z \neq x \ast (y \ast z)$ so the operation is not associative.

(b) For any three real numbers, $x$, $y$ and $z$,

$$x \odot (y \ast z) = x(y^2 + z^2) = xy^2 + xz^2$$

$$x \odot y \ast (x \odot z) = xy \ast xz = x^2y^2 + x^2z^2$$

$\therefore$ ($x \odot y) \ast (x \odot z) \neq x \odot (y \ast z)$

so the operation $\odot$ is not distributive over the operation $\ast$. 
Closed sets

A set is closed under an operation * when for any two members of the set, \( a * b \) gives another member of the set.

For example, the set of integers, \( \mathbb{Z} \), is closed under addition because for any two integers, \( a \) and \( b \), \( a + b \) is also an integer.

However, \( \mathbb{Z} \) is not closed under division because \( a \div b \) does not always give an integer, for example \( 3 \div 4 = \frac{3}{4} \), which is not an integer.

Identity

If \( a \) is any member of a set and there is one member \( b \) of the set, such that under an operation, \(*\), \( a * b = b * a = a \) then \( b \) is called the identity member of the set under the operation.

For example, 0 is the identity for members of \( \mathbb{R} \) under addition as, for any member \( a \),

\[
0 + a = a + 0 = a
\]

However, there is no identity for members of \( \mathbb{R} \) under subtraction because there is no real number \( b \) such that \( a - b = b - a \)

Also 1 is the identity for members of \( \mathbb{R} \) under multiplication, as for any member \( a \),

\[
1 	imes a = a 	imes 1 = a
\]

However, there is no identity for members of \( \mathbb{R} \) under division because there is no real number \( b \) such that \( a \div b = b \div a \)

Inverse

Any member \( a \) of a set has an inverse under an operation if there is another member of the set, which when combined with \( a \) gives the identity.

Clearly, a member can have an inverse only if there is an identity under the operation.

For example, as 0 is the identity for members of \( \mathbb{R} \) under addition, then \(-a\) is the inverse of any member \( a \), since \(-a + a = a + (-a) = 0\)

Also, as 1 is the identity for members of \( \mathbb{R} \) under multiplication, then \( \frac{1}{a} \) is the inverse of \( a \) since \( a \times \frac{1}{a} = 1 \),

but there is one important exception to this:

\[
\frac{1}{0} \text{ is meaningless.}
\]
Example
An operation, $\ast$, is defined for all real numbers $x$ and $y$ as

$$x \ast y = \frac{(x + y)}{2}$$

(a) Show that the set is closed under the operation $\ast$.
(b) Show that $x$ has no inverse under the operation $\ast$.

(a) When $x$ and $y$ are real numbers, $\frac{(x + y)}{2}$ is also a real number.
Therefore the set is closed under the operation $\ast$.

(b) For $x$ to have an inverse, there needs to be an identity member, i.e. one member, $b$, of $\mathbb{R}$ such that $x \ast b = x$,
i.e. such that $\frac{(x + b)}{2} = x$
Solving this for $b$ gives $b = x$
Now $x$ is any member of $\mathbb{R}$, so $b$ is not a single member and therefore there is no identity member of the set.
As there is no identity, $x$ has no inverse.

Exercise 1.2

1 Determine whether addition is distributive over multiplication on the set of real numbers.

2 The operation $\ast$ is given by
$$x \ast y = x^2y$$
for all real values of $x$ and $y$.
Determine whether the operation $\ast$ is:
(a) commutative
(b) associative
(c) distributive over addition.

3 The operation $\ast$ is given by
$$x \ast y = \sqrt{xy}$$
for all positive real numbers including 0.
(a) Show that the operation $\ast$ is closed.
(b) Write down the identity member.
(c) Determine whether each member has an inverse.

4 The operation $\sim$ is given by $x \sim y = \text{the difference between } x \text{ and } y$
for $x, y \in \mathbb{R}$.
(a) Determine whether $\mathbb{R}$ is closed under this operation.
(b) Show that the identity member is 0.
(c) Show that each member is its own inverse.
Surds

The square roots of most positive integers and fractions cannot be expressed exactly as either a fraction or as a terminating decimal, i.e. they are not rational numbers.

A number such as \( \sqrt{2} \) is an irrational number and can only be expressed exactly when left as \( \sqrt{2} \). In this form it is called a surd.

Note that \( \sqrt{2} \) means the positive square root of 2.

Simplifying surds

Many surds can be simplified.

For example, \( \sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2} \)

And \( \sqrt{8} + \sqrt{2} = \sqrt{4 \times 2} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} \)

In both cases, \( 3\sqrt{2} \) is the simplest possible surd form.

When a calculation involves surds, you should give your answer in the simplest possible surd form.

Operations on surds

An expression such as \( (3 - \sqrt{2})(2 - \sqrt{3}) \) can be expanded,

i.e. \( (3 - \sqrt{2})(2 - \sqrt{3}) = 6 - 3\sqrt{3} - 2\sqrt{2} + \sqrt{6} \)

|\( \sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} \)

When the same surd occurs in each bracket the expansion can be simplified.

For example,

\[
(5 - 2\sqrt{3})(3 + 2\sqrt{3}) = 15 - 6\sqrt{3} + 10\sqrt{3} - 12
\]

\[
\begin{align*}
2\sqrt{3} \times 2\sqrt{3} &= 4\sqrt{9} = 12 \\
&= 3 + 4\sqrt{3}
\end{align*}
\]

In particular, expressions of the form \( (a - \sqrt{b})(a + \sqrt{b}) \) simplify to a single rational number.

For example,

\[
(5 - 2\sqrt{3})(5 + 2\sqrt{3}) = 5^2 - (2\sqrt{3})^2
\]

\[
\begin{align*}
(2\sqrt{3})^2 &= 4\sqrt{3^2} = 4\sqrt{9} = 4 \times 3 \\
&= 25 - 12 = 13
\end{align*}
\]

Example

Simplify \( (2 - \sqrt{5})(3 + 2\sqrt{5}) \)

\[
(2 - \sqrt{5})(3 + 2\sqrt{5}) = 6 - 3\sqrt{5} + 4\sqrt{5} - 10
\]

\[
(2\sqrt{5} \times \sqrt{5} = 2\sqrt{25} = 10)
\]

\[
= -4 + \sqrt{5}
\]
Rationalising the denominator

When a fraction has a surd in the denominator, it can be transferred to the numerator.

When the denominator is a single surd, multiplying the fraction, top and bottom, by that surd will change the denominator into a rational number.

For example,

\[
\frac{2 + \sqrt{3}}{\sqrt{5}} = \frac{2 + \sqrt{3}}{\sqrt{5}} \times \sqrt{5} \sqrt{5} = \frac{2\sqrt{5} + \sqrt{15}}{5}
\]

When the denominator is of the form \(a + \sqrt{b}\), multiplying the fraction, top and bottom, by \(a - \sqrt{b}\) will change the denominator into a rational number.

For a denominator of the form \(a - \sqrt{b}\) multiply top and bottom by \(a + \sqrt{b}\).

Example

Rationalise the denominator and simplify \(\frac{\sqrt{2} - 1}{\sqrt{3}(\sqrt{2} + 3)}\)

This fraction has a single surd and a bracket in the denominator. Do not attempt to rationalise them both at the same time. We will start with rationalising the single surd.

\[
\frac{\sqrt{2} - 1}{\sqrt{3}(\sqrt{2} + 3)} = \frac{\sqrt{3}(\sqrt{2} - 1)}{\sqrt{3} \times \sqrt{3}(\sqrt{2} + 3)} = \frac{\sqrt{6} - \sqrt{3}}{3(\sqrt{2} + 3)}
\]

\[
= \frac{\sqrt{6} - \sqrt{3}}{3(\sqrt{2} + 3)} \times \frac{\sqrt{2} - 3}{\sqrt{2} - 3} = \frac{\sqrt{12} - \sqrt{6} - 3\sqrt{6} + 3\sqrt{3}}{3(\sqrt{2}^2 - 9)}
\]

\[
= \frac{2\sqrt{3} - 4\sqrt{6} + 3\sqrt{3}}{3(2 - 9)} = \frac{4\sqrt{6} - 5\sqrt{3}}{21}
\]

We have written down every step in this example, but you should be able to do some of these steps in your head.

Exercise 1.3

Expand and simplify when possible.

1. \((3 - 2\sqrt{3})(\sqrt{3} - \sqrt{2})\)
2. \((\sqrt{2} - \sqrt{5})^2\)
3. \((1 - (\sqrt{3} + \sqrt{2}))^2\)

Rationalise the denominator of each surd and simplify when possible.

4. \(\frac{2}{\sqrt{2}}\)
5. \(\frac{2\sqrt{2}}{\sqrt{3}}\)
6. \(\frac{1}{3 - \sqrt{2}}\)
7. \(\frac{1 - \sqrt{2}}{1 + \sqrt{2}}\)
8. \(\frac{\sqrt{3}}{2\sqrt{3} - 5\sqrt{5}}\)
9. \(\frac{\sqrt{8}}{\sqrt{2}(\sqrt{3} - \sqrt{2})}\)
Learning outcomes

- To identify simple and compound propositions
- To establish the truth value of compound statements using truth tables
- To state the converse, contrapositive and inverse of a conditional (implication) statement
- To determine whether two statements are logically equivalent

You need to know

- The meaning of commutative, distributive and associative for binary operations

Propositions

A sentence such as ‘Sonia went to school today’ is a closed sentence, but ‘She went to school today’ is not closed because it contains the variable ‘she’, who could be any female.

Closed sentences are called statements or propositions and are denoted by \( p \), \( q \), etc.

A proposition is either true or false.

Negation

The proposition ‘It is not raining’ contradicts the proposition ‘It is raining’.

‘It is not raining’ is called the negation of ‘It is raining’.

If \( p \) is the proposition ‘It is raining’, the negation of \( p \) is denoted by \( \sim p \).

Truth tables

For the proposition \( p \): ‘It is raining’, if \( p \) is true then \( \sim p \) is false.

But if \( p \) is false, then \( \sim p \) is true.

We can show this logic in table form (called a truth table).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \sim p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>0</td>
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</tr>
</tbody>
</table>

We use 1 to represent true and 0 to represent false.

The numbers in each column are called the truth values.

Conjunction

The statements \( p \): ‘It is raining’ and \( q \): ‘It is cold’ can be combined as ‘It is raining and it is cold’. This is called a conjunction of two propositions.

Using the symbol \( \land \) to mean ‘and’ we write this conjunction as \( p \land q \).

We can construct a truth table for \( p \land q \)

\( p \) can be true or false, \( q \) can also be true or false. We put all possible combinations of 1 (true) and 0 (false) for \( p \) and \( q \) in the first two columns. Then, reading across, we can complete the third column for \( p \land q \).

(If either \( p \) or \( q \) is false, then \( p \) and \( q \) must be false.)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Disjunction

The statements \( p \): ‘It is raining’ and \( q \): ‘It is cold’ can be combined as ‘It is raining or it is cold’. This is called a disjunction of two propositions and the word ‘and’ is implied so it would normally be written as ‘It is raining or it is cold.’

Using the symbol \( \lor \) to mean ‘or’ we write this disjunction as \( p \lor q \)
We can construct a truth table for $p \lor q$

$p$ can be true or false, $q$ can also be true or false. As before, we put all possible combinations of 1 and 0 for $p$ and $q$ in the first two columns. Then reading across we can complete the third column for $p \lor q$. (If either $p$ or $q$ or both are true, then $p$ or $q$ must be true.)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0</td>
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</tr>
</tbody>
</table>

**Conditional statements**

If ‘it is raining’ then ‘it is cold’ is called a *conditional statement*.

Using the symbol $\rightarrow$ to mean ‘If ... then ...’ we write $p \rightarrow q$

The proposition $p$ is called the *hypothesis* and the proposition $q$ is called the *conclusion*.

In logic, $p \rightarrow q$ is true except when a true hypothesis leads to a false conclusion.

For example, if $p$ is ‘5 is a prime number’ and $q$ is ‘6 is a prime number’ then in logic $p \rightarrow q$ is false.

The truth table for $p \rightarrow q$ is such that $p \rightarrow q$ is false for only one combination of $p$ and $q$: $p$ true and $q$ false.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
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</tbody>
</table>

The *converse* of $p \rightarrow q$ is $q \rightarrow p$

For example, the converse of ‘5 is a prime number’ $\rightarrow$ ‘6 is a prime number’ is ‘6 is a prime number’ $\rightarrow$ ‘5 is a prime number’.

Also the converse of ‘It is cold’ $\rightarrow$ ‘It is raining’ is ‘It is raining’ $\rightarrow$ ‘It is cold’.

The *inverse* of $p \rightarrow q$ is $\sim p \rightarrow \sim q$

For example the inverse of ‘5 is a prime number’ $\rightarrow$ ‘6 is a prime number’ is ‘5 is not a prime number’ $\rightarrow$ ‘6 is not a prime number’.

and the inverse of ‘It is raining’ $\rightarrow$ ‘It is cold’ is ‘It is not raining’ $\rightarrow$ ‘It is not cold’.

The *contrapositive* of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

For example the contrapositive of ‘5 is a prime number’ $\rightarrow$ ‘6 is a prime number’ is ‘6 is not a prime number’ $\rightarrow$ ‘5 is not a prime number’.

**Bi-conditional statements**

A *bi-conditional statement* is the conjunction of the conditional statement $p \rightarrow q$ with its converse $q \rightarrow p$, that is $(p \rightarrow q) \land (q \rightarrow p)$. This reads ‘if $p$ then $q$ and if $q$ then $p$’.

For example, ‘If it is raining then it is cold’ and ‘If it is cold then it is raining’ is a bi-conditional statement.

‘If it is raining then it is cold’ and ‘If it is cold then it is raining’ can be written simply as ‘It is raining’ if and only if ‘It is cold’.’
Using the symbol ⇔ to mean ‘if and only if’ we can write ‘It is raining’ ⇔ ‘It is cold’ and \((p \rightarrow q) \land (q \rightarrow p)\) can be written as \(p ⇔ q\).

We can construct a truth table for \(p ⇔ q\):

Start with the truth table for \(p \rightarrow q\), then add a column for \(q \rightarrow p\). Lastly, add a column for the conjunction of the third and fourth columns.

This table can now be written as a simpler truth table for a bi-conditional statement.

**Compound statements**

A compound statement combines two or more propositions using a combination of two or more of the symbols \(\sim, \land, \lor, \rightarrow, \leftarrow\).

A bi-conditional statement, \((p \rightarrow q) \land (q \rightarrow p)\), is an example of a compound statement.

---

**Example**

Let \(p\), \(q\) and \(r\) be the propositions:

- \(p\): ‘Students play soccer’, \(q\): ‘Students play cricket’, \(r\): ‘Students play basketball’.

Express the compound statement ‘Students play soccer or basketball but not both and students play cricket’ in symbolic form.

‘Students play soccer or basketball’ is \(p \lor r\). ‘Students do not play both soccer and basketball’ is \(\sim(p \land r)\).

‘Students play soccer or basketball but not both’ is \((p \lor r) \land \sim(p \land r)\).

Adding ‘and students play cricket’ to this gives \((p \lor r) \land \sim(p \land r) \land q\).

The truth table for a compound statement can be constructed in a similar way to the bi-conditional table above.

---

**Example**

Construct a truth table for the compound statement \(p \lor (\sim q \land p)\)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(\sim q)</th>
<th>(\sim q \land p)</th>
<th>(p \lor (\sim q \land p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0</td>
<td>0</td>
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Always start with \(p\) and \(q\). Then add columns in stages to build up the compound statement.
Equivalence

Two statements are logically equivalent when their truth values are the same, that is in the completed truth tables the final columns are identical.

Example

Determine whether the statements $p \land q$ and $\sim p \rightarrow q$ are logically equivalent.

We construct a truth table for each statement:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\sim p$</th>
<th>$p \land q$</th>
<th>$\sim p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
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</tbody>
</table>

The truth values for $p \land q$ and $\sim p \rightarrow q$ are not the same. Therefore the statements are logically not equivalent. We write $p \land q \neq \sim p \rightarrow q$.

Identity law

This law states that $p \land p$ and $p \lor p$ are both equivalent to $p$.

Algebra of propositions

The symbols $\land$ and $\lor$ are called logical connectors.

These connectors are commutative, that is $p \land q = q \land p$ and $p \lor q = q \lor p$.

They are also associative, that is $(p \land q) \land r = p \land (q \land r)$.

They are also distributive over each other and over the conditional $\rightarrow$, for example

$p \land (q \lor r) = (p \land q) \lor (p \land r)$ and $p \lor (q \land r) = (p \lor q) \land (p \lor r)$

$p \land (q \rightarrow r) = (p \land q) \rightarrow (p \land r)$ and $p \lor (q \rightarrow r) = (p \lor q) \rightarrow (p \lor r)$

These properties can be proved using truth tables.

The properties can also be used to prove the equivalence between two compound statements.

It can also be shown that $p \rightarrow q = \sim q \rightarrow \sim p$.

Exercise 1.4

In this exercise, $p$, $q$ and $r$ are propositions.

1. Write down the contrapositive of $\sim p \land q$.

2. (a) Construct a truth table for $p \rightarrow \sim q$ and $p \lor \sim q$.

   (b) State, with a reason, whether $p \rightarrow \sim q$ and $p \lor \sim q$ are logically equivalent.


   Using logic symbols, write down in terms of $p$, $q$ and $r$ the statement: ‘The sun is shining and it is cold and it is not raining’.
### 1.5 Direct proof

**Learning outcomes**
- To construct simple proofs, specifically direct proofs
- Proof by the use of counter examples

**You need to know**
- The basic rules of logic
- How to solve a quadratic equation by factorisation or by the formula
- How to find the area of a triangle

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**Direct proof**

Mathematics is the study of numbers, shapes, space and change. Mathematicians look for patterns and formulate conjectures. They then try to prove the truth, or otherwise, of conjectures by proof that is built up from axioms. The axioms are the basic rules or definitions, and all other facts can be derived from these by deduction, that is by using true inferences from those rules (an inference is the same as the logic conditional $\rightarrow$). (We can use the game of chess as an analogy – the basic rules are the moves that are allowed for each piece, and games are built up from these moves.)

For example, $x^n$ is defined to mean $n$ lots of $x$ multiplied together, i.e. $x \times x \times x \times \ldots \times x$, and from this definition we can deduce that $x^a \times x^b = x^{a+b}$.

#### Example

Prove that if $4(x - 5) = 8$ then $x = 7$

Starting with $4(x - 5) = 8$

$\Rightarrow$ $4x - 20 = 8$ Using the distributive law

$\Rightarrow$ $4x = 28$ Adding 20 to each side keeps the equality true

$\Rightarrow$ $x = 7$ Dividing each side by 4 keeps the equality true

$\therefore$ $4(x - 5) = 8 \Rightarrow x = 7$

This is an example of direct proof by deduction, i.e. to prove $p \Rightarrow q$, start with $p$ then deduce $p \Rightarrow r \Rightarrow s \Rightarrow q$, so $p \Rightarrow q$.

(Note that this is a proof of an implication $p \Rightarrow q$. We know from Topic 1.4 that whether $p$ is true or $q$ is true is another question.)

We know from logic that if $p \rightarrow q$ is true then the contrapositive $\neg q \rightarrow \neg p$ is also true. Therefore we can also say that if $x \neq 7$ then $4(x - 5) \neq 8$.

The converse of $4(x - 5) = 8 \Rightarrow x = 7$, i.e. $x = 7 \Rightarrow 4(x - 5) = 8$ is also true in this case, but the converse of a true implication is not always true.

For example, ‘A polygon is a square $\Rightarrow$ a polygon has four equal sides’ is true but ‘A polygon has four equal sides $\Rightarrow$ a polygon is a square’ is not true because a rhombus has four equal sides.

Therefore the converse of an implication needs to be proved to be true.

---

**Example**

Prove that the sum of the interior angles of any triangle is 180°.

$\forall ABC$ is any triangle, $DE$ is parallel to $AB$.

$\angle DCA = \angle CAB$ Alternate angles are equal

$\angle ECB = \angle CBA$ Alternate angles are equal

$\angle DCA + \angle ACB + \angle ECB = 180°$ Supplementary angles

$\Rightarrow \angle CAB + \angle ACB + \angle ECB = 180°$

$\Rightarrow$ the sum of the interior angles of any triangle is 180°.
Use of counter examples

As well as it being necessary to prove that a statement is true, it is also important to prove that a statement is false. This is particularly important for the converse of a true implication.

A statement can be shown to be false if we can find just one example that disproves it. This is called a counter example.

For example, \( a > 0 \Rightarrow a^2 > 0 \) is true, but the converse \( a^2 > 0 \Rightarrow a > 0 \) is false.

We can use \( a^2 = 9 \Rightarrow a = 3 \) or \(-3\) as a counter example because \(-3 \not> 0\).

For example, the statement ‘all prime numbers are odd’ is not true. We can prove this using the counter example ‘2 is a prime number and 2 is not an odd number’.

**Example**

Use a counter example to prove that the converse of the true statement: ‘\( n \) is an integer’ \( \Rightarrow \) ‘\( n^2 \) is an integer’ is false.

The converse of the given statement is ‘\( n^2 \) is an integer’ \( \Rightarrow \) ‘\( n \) is an integer’.

\( (\sqrt{2})^2 = 2 \) is an integer but \( \sqrt{2} \) is not an integer.

Therefore ‘\( n^2 \) is an integer’ \( \Rightarrow \) ‘\( n \) is an integer’ is false.

**Exercise 1.5**

1. Prove that if \( x^2 - 3x + 2 = 0 \) then \( x = 1 \) or \( x = 2 \)
2. Find a counter example to show that \( a > b \Rightarrow a^2 > b^2 \) is not true.
3. (a) Prove that ‘\( n \) is an odd integer \( \Rightarrow \) \( n^2 \) is an odd integer’.
   (Start with \( n = 2k + 1 \) where \( k \) is any integer.)
   (b) Use a counter example to show that the converse of the statement in (a) is false.
4. (a) Prove that ‘\( x^2 + bx + c = 0 \) has equal roots \( \Rightarrow b^2 = 4c \)’
   (b) Prove that the converse of the statement in (a) is also true.
5. In the diagram, D is the midpoint of AB.
   Prove that the area of triangle ABC is twice the area of triangle ADC.
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- **The key terms** you need to know
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