Fractions

Unit fractions

Fractions have been used for a very long time.

The Ancient Egyptians wrote all fractions with 1 as the numerator. These are called unit fractions. They did not use fractions such as $\frac{5}{8}$

They used combinations of fractions such as $\frac{1}{2} + \frac{1}{4}$

Use the fraction button on your calculator to work out how Ancient Egyptians would write $\frac{11}{12}$, $\frac{7}{10}$, and $\frac{7}{12}$

After the work in this chapter you should not need a calculator to do this.

4.1 Comparing fractions

Equivalent fractions

The fractions $\frac{1}{4}$, $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$ are equivalent to each other. They are called equivalent fractions.

You can multiply the numerator and denominator by the same number to obtain an equivalent fraction.

$$\frac{1}{4} \times 3 = \frac{3}{12}$$

You can divide the numerator and denominator by the same number.

$$\frac{4}{16} \div 4 = \frac{1}{4}$$

This is called simplifying.

When a fraction cannot be simplified further it is said to be in its simplest form or lowest terms.
Comparing and ordering fractions

It is easy to see that \( \frac{6}{8} \) is greater than \( \frac{5}{8} \).

They have the same denominator so you can just compare the numerators to see which is the larger fraction.

It is not so easy to see whether \( \frac{6}{8} \) is greater or less than \( \frac{2}{3} \).

You need to change them to fractions that have the same denominator.

\[
\frac{6}{8} = \frac{18}{24} \\
\frac{2}{3} = \frac{16}{24} \\
\]

\( \frac{6}{8} \) is greater than \( \frac{2}{3} \), or \( \frac{6}{8} > \frac{2}{3} \).

Remember that \( > \) means ‘is greater than’ and \( < \) means ‘is less than’.

**Worked example 1**

Simplify \( \frac{28}{32} \).

Give your answer in its lowest terms.

\[
\frac{28}{32} \div \frac{2}{2} = \frac{14}{16} = \frac{7}{8} \div \frac{2}{2}
\]

This could be done in one step by dividing the numerator and denominator by 4.

\( \frac{28}{32} = \frac{7}{8} \)

This cannot be simplified further. It is in its lowest terms.

**Worked example 2**

Write these fractions in order of size, smallest first.

\( \frac{5}{12} \), \( \frac{3}{8} \), \( \frac{1}{3} \)

Change these to fractions that have the same denominator.

This is the lowest common multiple of 3, 8 and 12.

This is 24.

\[
\frac{5}{12} \times \frac{2}{2} = \frac{10}{24} \\
\frac{3}{8} \times \frac{3}{3} = \frac{9}{24} \\
\frac{1}{3} \times \frac{8}{8} = \frac{8}{24}
\]

It is now easy to write the fractions in order of size.

\( \frac{8}{24} \), \( \frac{9}{24} \), \( \frac{10}{24} \)

The answer to the question is \( \frac{1}{3} \), \( \frac{3}{8} \), \( \frac{5}{12} \).
Exercise 4.1

1 Copy and complete the following equivalent fractions.
   a \( \frac{7}{8} = \frac{21}{24} \)  
   b \( \frac{5}{6} = \frac{15}{18} \)
   c \( \frac{7}{10} = \frac{28}{40} \)
   d \( \frac{15}{18} = \frac{5}{6} \)
   e \( \frac{18}{24} = \frac{3}{4} \)
   f \( \frac{9}{24} = \frac{3}{8} \)

2 There are three sets of equivalent fractions here.
   They have been mixed up.
   Sort them into the three sets.
   \( \frac{8}{24}, \frac{10}{24}, \frac{4}{12}, \frac{8}{20}, \frac{3}{12}, \frac{5}{20}, \frac{6}{24}, \frac{6}{15}, \frac{5}{15} \)

3 Write each of these fractions in their lowest terms.
   a \( \frac{14}{20} \)
   b \( \frac{16}{24} \)
   c \( \frac{15}{45} \)
   d \( \frac{24}{60} \)
   e \( \frac{24}{84} \)
   f \( \frac{18}{48} \)

4 Which fraction in each pair is greater?
   a \( \frac{5}{7}, \frac{3}{5} \)
   b \( \frac{4}{7}, \frac{2}{3} \)
   c \( \frac{15}{10}, \frac{2}{7} \)
   d \( \frac{2}{3}, \frac{5}{8} \)
   e \( \frac{3}{5}, \frac{7}{11} \)
   f \( \frac{3}{8}, \frac{2}{5} \)

5 Complete each statement with the correct sign <, = or >.
   a \( \frac{3}{4} \bigg\{\frac{11}{12} \)
   b \( \frac{5}{7} \bigg\{\frac{2}{3} \)
   c \( \frac{11}{16} \bigg\{\frac{3}{4} \)
   d \( \frac{9}{15} \bigg\{\frac{3}{5} \)
   e \( \frac{7}{9} \bigg\{\frac{4}{5} \)
   f \( \frac{2}{3} \bigg\{\frac{11}{16} \)

6 Adil, Elena and Billy share two identical bags of sweets between them.
   Adil keeps \( \frac{4}{5} \) of the sweets in the first bag and gives the rest to Elena.
   Billy keeps \( \frac{2}{3} \) of the sweets in the second bag and gives the rest to Elena.
   a Who has the greatest number of sweets?
   b Who has the least number of sweets?
   Show how you work out your answers.

7 a I am equivalent to \( \frac{1}{2} \)
   The sum of my numerator and denominator is 21.
   What fraction am I?
   b I am equivalent to \( \frac{3}{5} \)
   The product of my numerator and denominator is 240.
   What fraction am I?
   c I am equivalent to \( \frac{3}{4} \)
   The difference between my numerator and denominator is 7.
   What fraction am I?
4.2 Addition and subtraction of fractions and mixed numbers

Addition

Fractions need to have the same denominator in order to be added together.

The method is:

1. Find the lowest common multiple of the denominators.
2. Write each fraction with this denominator.
3. Add the two fractions.
4. Simplify the answer if possible.

When adding mixed numbers it is best to add the integer and fraction parts separately.

Worked example 1

Work out $\frac{1}{3} + \frac{1}{4}$

The lowest common multiple of 3 and 4 is 12.

$\frac{1}{3} \times 4 = \frac{4}{12}$
$\frac{1}{4} \times 3 = \frac{3}{12}$

$\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

Worked example 2

Work out $\frac{5}{6} + \frac{7}{8}$

The lowest common multiple of 6 and 8 is 24.

$\frac{5}{6} \times 4 = \frac{20}{24}$
$\frac{7}{8} \times 3 = \frac{21}{24}$

$\frac{5}{6} + \frac{7}{8} = \frac{20}{24} + \frac{21}{24} = \frac{41}{24} = 1\frac{17}{24}$

Worked example 3

Work out $1\frac{1}{4} + 2\frac{1}{3}$

$1\frac{1}{4} + 2\frac{1}{3} = 1 + 2 + \frac{1}{4} + \frac{1}{3}$

$= 3 + \frac{3}{12} + \frac{4}{12}$

$= 3 + \frac{7}{12}$

$= 3\frac{7}{12}$
Subtraction

This is done in almost exactly the same way as for addition.

The method is:

1. Find the lowest common multiple of the denominators.
2. Write each fraction with this denominator.
3. Subtract the two fractions.
4. Simplify the answer if possible.

When subtracting mixed numbers it is best to write them as improper fractions first.

Worked example 4

Work out $\frac{3}{4} - \frac{2}{5}$

The lowest common multiple of 4 and 5 is 20.

$\frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$
$\frac{2}{5} \times \frac{4}{4} = \frac{8}{20}$

$\frac{3}{4} - \frac{2}{5} = \frac{15}{20} - \frac{8}{20} = \frac{7}{20}$

Worked example 5

Work out $1\frac{1}{2} - \frac{4}{5}$

Write mixed numbers as improper fractions:

$1\frac{1}{2} - \frac{4}{5} = \frac{3}{2} - \frac{4}{5}$

The lowest common multiple of 2 and 5 is 10.

$\frac{3}{2} \times \frac{5}{5} = \frac{15}{10}$
$\frac{4}{5} \times \frac{2}{2} = \frac{8}{10}$

$\frac{3}{2} - \frac{4}{5} = \frac{15}{10} - \frac{8}{10} = \frac{7}{10}$

Worked example 6

Work out $3\frac{1}{4} - 1\frac{2}{5}$

Write mixed numbers as improper fractions:

$3\frac{1}{4} - 1\frac{2}{5} = \frac{13}{4} - \frac{7}{5}$

The lowest common multiple of 4 and 5 is 20.

$\frac{13}{4} \times \frac{5}{5} = \frac{65}{20}$
$\frac{7}{5} \times \frac{4}{4} = \frac{28}{20}$

Remember that a fraction is an improper fraction if the numerator is larger than the denominator.
Worked example 7

When $\frac{2}{5}$ is added to a fraction the answer is $\frac{11}{15}$. What is the fraction?

The fraction is $\frac{11}{15} - \frac{2}{5}$

$= \frac{11}{15} - \frac{6}{15}$

$= \frac{5}{15} = \frac{1}{3}$

The inverse of addition is subtraction.

Worked example 8

When $\frac{2}{7}$ is subtracted from a fraction the answer is $\frac{8}{21}$. What is the fraction?

The fraction is $\frac{8}{21} + \frac{2}{7}$

$= \frac{8}{21} + \frac{6}{21}$

$= \frac{14}{21} = \frac{2}{3}$

The inverse of subtraction is addition.

Exercise 4.2

1 Work these out.

Simplify your answer where possible.

a) $\frac{5}{8} + \frac{1}{8}$  

b) $\frac{2}{7} + \frac{3}{7}$

c) $\frac{5}{6} + \frac{1}{6}$

d) $\frac{7}{10} - \frac{3}{10}$

e) $\frac{7}{9} - \frac{1}{9}$

f) $\frac{4}{7} - \frac{1}{7}$

2 Work these out.

Simplify your answer where possible.

a) $\frac{1}{2} + \frac{1}{4}$

b) $\frac{1}{6} + \frac{5}{12}$

c) $\frac{5}{16} + \frac{3}{8}$

d) $\frac{7}{8} - \frac{1}{4}$

e) $\frac{2}{3} - \frac{1}{6}$

f) $\frac{3}{5} - \frac{3}{10}$

3 Work these out.

Simplify your answer where possible.

a) $\frac{2}{3} + \frac{1}{4}$

b) $\frac{2}{5} + \frac{1}{3}$

c) $\frac{1}{4} + \frac{2}{5}$

d) $\frac{1}{3} - \frac{2}{7}$

e) $\frac{3}{8} - \frac{1}{5}$

f) $\frac{1}{6} - \frac{1}{8}$

4 Work these out.

Simplify your answer where possible.

a) $\frac{4}{5} - \frac{1}{3}$

b) $\frac{3}{5} - \frac{1}{4}$

c) $\frac{7}{8} - \frac{2}{3}$

d) $\frac{5}{6} - \frac{1}{5}$

e) $\frac{7}{10} - \frac{1}{3}$

f) $\frac{7}{10} - \frac{2}{15}$
5 Work these out.
Simplify your answer where possible.
\[
\begin{align*}
\text{a} & \quad \frac{3}{4} + \frac{2}{3} \quad \text{b} & \quad \frac{5}{6} + \frac{3}{8} \quad \text{c} & \quad \frac{5}{7} + \frac{4}{5} \\
\text{d} & \quad \frac{11}{12} - \frac{3}{8} \quad \text{e} & \quad \frac{9}{10} - \frac{5}{6} \quad \text{f} & \quad \frac{2}{3} - \frac{3}{16}
\end{align*}
\]

6 When \( \frac{1}{3} \) is added to a fraction the answer is \( 1\frac{1}{12} \). What is the fraction?

7 When \( \frac{3}{5} \) is subtracted from a fraction the answer is \( \frac{1}{15} \). What is the fraction?

8 a What do the fractions on the top row of the grid add up to?

\[
\begin{array}{ccc}
\frac{1}{3} & \frac{1}{8} & \frac{1}{6} \\
\frac{\ }{ } & \frac{ }{ } & \frac{3}{8} \\
\frac{ }{ } & \frac{ }{ } & \frac{ }{ }
\end{array}
\]

b Copy the grid.
Fill in the gaps to make a magic square.
Every row, every column and the two diagonals must have the same total.

9 Work these out.
Simplify your answer where possible.
\[
\begin{align*}
\text{a} & \quad \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \\
\text{b} & \quad \frac{1}{3} + \frac{1}{5} - \frac{1}{6}
\end{align*}
\]

10 Work out
\[
\begin{align*}
\text{a} & \quad \frac{1\frac{1}{4}}{ } + \frac{1}{3} \\
\text{b} & \quad \frac{1}{6} + 2\frac{1}{4} \\
\text{c} & \quad \frac{1\frac{1}{3}}{ } + 1\frac{1}{5} \\
\text{d} & \quad 2\frac{2}{5} + 1\frac{1}{4}
\end{align*}
\]

11 Work out
\[
\begin{align*}
\text{a} & \quad 2\frac{2}{3} - 1\frac{1}{6} \\
\text{b} & \quad \frac{1\frac{4}{5}}{ } - \frac{1}{3} \\
\text{c} & \quad 3\frac{3}{8} - 1\frac{1}{4} \\
\text{d} & \quad 2\frac{5}{8} - 1\frac{1}{3}
\end{align*}
\]

12 Work these out.
Give your answers in their lowest terms.
\[
\begin{align*}
\text{a} & \quad \frac{1\frac{1}{2}}{ } + 2\frac{1}{3} \\
\text{b} & \quad \frac{2\frac{3}{4}}{ } + 1\frac{7}{8} \\
\text{c} & \quad 3\frac{2}{5} + 1\frac{3}{4} \\
\text{d} & \quad 2\frac{\cancel{1}}{\cancel{2}} - \frac{3}{4} \\
\text{e} & \quad 1\frac{1}{2} - \frac{2}{3} \\
\text{f} & \quad 2\frac{1}{4} - 1\frac{4}{5}
\end{align*}
\]

13 This fraction sum is made from four different digits 1, 2, 4 and 8.
The fraction sum is 1.
\[
\frac{1}{2} + \frac{4}{8}
\]
Find other fraction sums. Use four different digits. Make sure the fraction sum is 1.

14 On four days each week Penny cycles for \( 1\frac{1}{4} \) hours.
On the other three days she walks for \( 2\frac{1}{2} \) hours and runs for \( \frac{3}{4} \) hour.
\[
\begin{align*}
\text{a} & \quad \text{How many hours does Penny exercise for each week?} \\
\text{b} & \quad \text{Penny decides to cycle for three days each week and walk and run on the other four days.}
\end{align*}
\]
Will the time she spends exercising increase or decrease?
By how many hours will it change?
4.3 Multiplying and dividing an integer by a fraction

Multiplying an integer by a fraction

Remember an integer is a whole number.

\[3 \times \frac{4}{5} \text{ means the same as } \frac{4}{5} + \frac{4}{5} + \frac{4}{5}, \text{ which is } \frac{4 + 4 + 4}{5} = \frac{3 \times 4}{5} = \frac{12}{5}\]

To multiply an integer by a fraction you multiply the numerator by the integer.

This gives you the numerator of your answer.

The denominator does not change.

**Worked example 1**

Work out \(7 \times \frac{1}{4}\)

\[7 \times \frac{1}{4} = \frac{7 \times 1}{4} = \frac{7}{4} = 1 \frac{3}{4}\]

**Finding a fraction of a quantity**

To find \(\frac{1}{4}\) of 5, you can think of this as:

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美景
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This is the same as \(5 \times \frac{1}{4}\)

\[= \frac{5}{4}\]

\[= 1 \frac{1}{4}\]

When you see ‘of’ in maths it usually means ‘\(\times\)’.

**Worked example 2**

Find \(\frac{1}{3}\) of 7.

\[\frac{1}{3}\text{ of } 7 = \frac{1}{3} \times 7 = \frac{7}{3} = 2 \frac{1}{3}\]
Dividing an integer by a fraction

When you work out $12 \div 2$ you are finding how many $2$s there are in $12$.
The same applies to fractions.
$2 \div \frac{1}{4}$ means ‘how many $\frac{1}{4}$’s are there in $2$?’
Remember the denominator tells you the number of parts each whole is divided into.

You can see from this diagram $2 \div \frac{1}{4} = 8$
There are four quarters in each ‘whole’.
In two ‘wholes’ there are $2 \times 4 = 8$ quarters
You can work out $2 \div \frac{1}{5}$ in the same way.

$2 \div \frac{1}{5} = 2 \times 5 = 10$
When you work out $2 \div \frac{2}{5}$ this fraction does not have $1$ as the numerator.
As $\frac{2}{5}$ is twice $\frac{1}{5}$ there will be half as many.
$2 \div \frac{2}{5} = 5$ as shown in this diagram.
This is worked out as $2 \times \frac{5}{2} = \frac{10}{2} = 5$

To divide an integer by a fraction, you turn the fraction ‘upside down’ and multiply by it.

**Worked example 3**

Work out $4 \div \frac{2}{3}$

$$4 \div \frac{2}{3} = 4 \times \frac{3}{2} = \frac{12}{2} = 6$$

**Exercise 4.3**

1. Work out
   - a) $2 \times \frac{3}{11}$
   - b) $2 \times \frac{2}{5}$
   - c) $4 \times \frac{2}{15}$
   - d) $7 \times \frac{2}{19}$
   - e) $5 \times \frac{3}{16}$
   - f) $3 \times \frac{2}{7}$

2. Work these out.
   Give your answers in their lowest terms.
   Write any improper fractions as mixed numbers.

   - a) $4 \times \frac{3}{8}$
   - b) $4 \times \frac{5}{7}$
   - c) $5 \times \frac{2}{3}$
   - d) $15 \times \frac{5}{6}$
   - e) $15 \times \frac{2}{5}$
   - f) $20 \times \frac{3}{4}$
   - g) $12 \times \frac{3}{4}$
   - h) $3 \times \frac{10}{27}$
   - i) $40 \times \frac{3}{8}$

3. Find
   - a) $\frac{1}{3}$ of 15
   - b) $\frac{1}{4}$ of 20
   - c) $\frac{2}{3}$ of 12
   - d) $\frac{3}{8}$ of 24
   - e) $\frac{2}{5}$ of 15
   - f) $\frac{3}{5}$ of 30

4. Find
   - a) $\frac{2}{3}$ of 5
   - b) $\frac{3}{4}$ of 7
   - c) $\frac{3}{8}$ of 20
   - d) $\frac{2}{5}$ of 17
   - e) $\frac{7}{8}$ of 12
   - f) $\frac{5}{6}$ of 15

5. Which of the following amounts is greater?
   - a) $\frac{3}{5}$ of 125 or $\frac{3}{8}$ of 240
   - b) $\frac{4}{5}$ of 130 or $\frac{2}{3}$ of 144

6. Work out
   - a) $2 \div \frac{1}{3}$
   - b) $3 \div \frac{1}{4}$
   - c) $5 \div \frac{1}{7}$
   - d) $6 \div \frac{1}{5}$
   - e) $4 \div \frac{1}{8}$
   - f) $7 \div \frac{1}{6}$

7. Work out
   - a) $2 \div \frac{2}{3}$
   - b) $3 \div \frac{3}{4}$
   - c) $5 \div \frac{5}{7}$
   - d) $4 \div \frac{2}{5}$
   - e) $6 \div \frac{3}{5}$
   - f) $14 \div \frac{7}{10}$
8 Work these out.

Give your answers in their lowest terms.

Write any improper fractions as mixed numbers.

\[ \text{a} \quad 6 \div \frac{3}{4} \quad \text{b} \quad 5 \div \frac{3}{4} \quad \text{c} \quad 3 \div \frac{2}{3} \]
\[ \text{d} \quad 5 \div \frac{3}{8} \quad \text{e} \quad 6 \div \frac{4}{5} \quad \text{f} \quad 7 \div \frac{2}{3} \]

9 Copy and complete these patterns.

\[ \text{a} \quad 60 \times \frac{1}{6} = 10 \quad \text{b} \quad 10 \div \frac{1}{6} = 60 \]
\[ 30 \times \frac{2}{6} = 10 \quad 10 \div \ldots \]
\[ 20 \times \frac{3}{6} = 10 \quad \ldots \]
\[ \ldots \]
\[ 12 \times \frac{5}{6} = 10 \quad \ldots \]

Review

1 Write these fractions in their lowest terms.

\[ \text{a} \quad \frac{8}{12} \quad \text{b} \quad \frac{9}{15} \quad \text{c} \quad \frac{7}{35} \]
\[ \text{d} \quad \frac{8}{18} \quad \text{e} \quad \frac{18}{60} \quad \text{f} \quad \frac{12}{30} \]

2 Which fraction in each pair is greater?

\[ \text{a} \quad \frac{1}{4}, \frac{3}{20} \quad \text{b} \quad \frac{2}{5}, \frac{3}{10} \quad \text{c} \quad \frac{14}{25}, \frac{7}{15} \]

3 Work these out.

Simplify your answer where possible.

\[ \text{a} \quad \frac{1}{4} + \frac{1}{3} \quad \text{b} \quad \frac{3}{4} + \frac{2}{3} \quad \text{c} \quad \frac{2}{3} - \frac{1}{8} \]
\[ \text{d} \quad \frac{5}{8} - \frac{1}{5} \quad \text{e} \quad \frac{7}{8} + \frac{2}{3} \quad \text{f} \quad \frac{5}{6} - \frac{3}{7} \]
\[ \text{g} \quad 3\frac{4}{5} + 2\frac{5}{8} \quad \text{h} \quad 3\frac{2}{9} - \frac{5}{6} \quad \text{i} \quad 2\frac{3}{8} - 1\frac{2}{3} \]

4 Work these out.

Do not simplify your answers.

\[ \text{a} \quad \frac{1}{2} + \frac{1}{4} \]
\[ \text{b} \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \]
\[ \text{c} \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \]
\[ \text{d} \quad \text{Your answers to a, b, and c are the start of a sequence. What are the next two terms in the sequence?} \]

5 Work these out.

Simplify your answer where possible.

Write any improper fractions as mixed numbers.

\[ \text{a} \quad 4 \times \frac{1}{5} \quad \text{b} \quad 7 \times \frac{3}{5} \quad \text{c} \quad \frac{3}{4} \times 36 \quad \text{d} \quad \frac{5}{7} \times 24 \]

6 Work out

\[ \text{a} \quad 7 \div \frac{1}{5} \quad \text{b} \quad 5 \div \frac{1}{8} \quad \text{c} \quad 3 \div \frac{3}{5} \quad \text{d} \quad 4 \div \frac{5}{8} \]