A note on academic honesty

It is of vital importance to acknowledge and appropriately credit the owners of information when that information is used in your work. After all, owners of ideas (intellectual property) have property rights. To have an authentic piece of work, it must be based on your individual and original ideas with the work of others fully acknowledged. Therefore, all assignments, written or oral, completed for assessment must use your own language and expression. Where sources are used or referred to, whether in the form of direct quotation or paraphrase, such sources must be appropriately acknowledged.

How do I acknowledge the work of others?

The way that you acknowledge that you have used the ideas of other people is through the use of footnotes and bibliographies.

Footnotes (placed at the bottom of a page) or endnotes (placed at the end of a document) are to be provided when you quote or paraphrase from another document, or closely summarize the information provided in another document. You do not need to provide a footnote for information that is part of a “body of knowledge”. That is, definitions do not need to be footnoted as they are part of the assumed knowledge.

Bibliographies should include a formal list of the resources that you used in your work. “Formal” means that you should use one of the several accepted forms of presentation. This usually involves separating the resources that you use into different categories (e.g. books, magazines, newspaper articles, Internet-based resources, CDs and works of art) and providing full information as to how a reader or viewer of your work can find the same information. A bibliography is compulsory in the extended essay.

What constitutes malpractice?

Malpractice is behavior that results in, or may result in, you or any student gaining an unfair advantage in one or more assessment component. Malpractice includes plagiarism and collusion.

Plagiarism is defined as the representation of the ideas or work of another person as your own. The following are some of the ways to avoid plagiarism:

- Words and ideas of another person used to support one’s arguments must be acknowledged.
- Passages that are quoted verbatim must be enclosed within quotation marks and acknowledged.
- CD-ROMs, email messages, web sites on the Internet, and any other electronic media must be treated in the same way as books and journals.
- The sources of all photographs, maps, illustrations, computer programs, data, graphs, audio-visual, and similar material must be acknowledged if they are not your own work.
- Works of art, whether music, film, dance, theatre arts, or visual arts, and where the creative use of a part of a work takes place, must be acknowledged.

Collusion is defined as supporting malpractice by another student. This includes:

- allowing your work to be copied or submitted for assessment by another student
- duplicating work for different assessment components and/or diploma requirements.

Other forms of malpractice include any action that gives you an unfair advantage or affects the results of another student. Examples include, taking unauthorized material into an examination room, misconduct during an examination, and falsifying a CAS record.

About the book

The new syllabus for Mathematics Standard Level is thoroughly covered in this book. Each chapter is divided into lesson size sections with the following features:

- **Investigations**
- **Examination suggestions**
- **Examiner’s tip**
- **Theory of Knowledge**
- **Did you know?**
- **Historical exploration**

Mathematics is a most powerful, valuable instrument that has both beauty in its own study and usefulness in other disciplines. The Sumerians developed mathematics as a recognized area of teaching and learning about 5,000 years ago and it has not stopped developing since then.

The Course Companion will guide you through the latest curriculum with full coverage of all topics and the new internal assessment. The emphasis is placed on the development and improved understanding of mathematical concepts and their real life application as well as proficiency in problem solving and critical thinking. The Course Companion denotes questions that would be suitable for examination practice and those where a GDC may be used. Questions are designed to increase in difficulty, strengthen analytical skills and build confidence through understanding. Internationalism, ethics and applications are clearly integrated into every section and there is a TOK application page that concludes each chapter.

It is possible for the teacher and student to work through in sequence but there is also the flexibility to follow a different order. Where appropriate the solutions to examples using the TI-Nspire calculator are shown. Similar solutions using the TI-84 Plus and Casio FX-9860GIIB are included on the accompanying interactive CD which includes a complete ebook of the text, prior learning, GDC support, an interactive glossary, sample examination papers, internal assessment support, and ideas for the exploration.

Mathematics education is a growing, ever changing entity. The contextual, technology integrated approach enables students to become adaptable, life-long learners.

Note: US spelling has been used, with IB style for mathematical terms.

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What’s on the CD?

The material on your CD-ROM includes the entire student book as an eBook, as well as a wealth of other resources specifically written to support your learning. On these two pages you can see what you will find and how it will help you to succeed in your Mathematics Standard Level course.

The whole print text is presented as a user-friendly eBook for use in class and at home.

Extra content can be found in the Contents menu or attached to specific pages.

This icon appears in the book wherever there is extra content.

Navigation is straightforward either through the Contents Menu, or through the Search and Go to page tools.

A range of tools enables you to zoom in and out and to annotate pages with your own notes.

The glossary provides comprehensive coverage of the language of the subject and explains tricky terminology. It is fully editable making it a powerful revision tool.

Extension material is included for each chapter containing a variety of extra exercises and activities. Full worked solutions to this material are also provided.

Glossary

<table>
<thead>
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<th>Term</th>
<th>Definition</th>
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<td>acute</td>
<td>$\frac{\pi}{4}$ radians or $45^\circ$</td>
</tr>
<tr>
<td>trapezium</td>
<td>A quadrilateral with one pair of parallel sides</td>
</tr>
<tr>
<td>obtuse</td>
<td>More than $90^\circ$ but less than $180^\circ$</td>
</tr>
</tbody>
</table>

An algebraic look at limits

Section 1.1 continued:

Basic limit properties:

1. If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.
2. If $f(x)$ is continuous at $x = a$, then $af(x)$ is continuous at $x = a$ for any constant $a$.
3. If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)g(x)$ is continuous at $x = a$.
4. If $f(x)$ is continuous at $x = a$, then $\frac{f(x)}{g(x)}$ is continuous at $x = a$, provided $g(a) \neq 0$.

Example 1

Find the limit of $f(x)$ as $x$ approaches 2.

Solution:

1. $\lim_{x \to 2} f(x) = 6$
2. $\lim_{x \to 2} g(x) = 4$
3. $\lim_{x \to 2} h(x) = 0$
Practice paper 2

A metal Petzal is designed in the shape of a cylinder with a rectangular base. The Petzal is 60 cm high. The base is 35 cm wide and 20 cm long. The Petzal has a mass of 950 kg.

1. Write down the length of Jan’s task.
2. Calculate the perimeter of the base of Jan’s task.
3. Calculate the height of Jan’s task.
4. Calculate the volume of Jan’s task.
5. A cube is designed so that its volume is minimized. To do this the length of the base of the cube is a, where a is measured in cm.
6. Write down the length of Kata’s task in terms of a.
7. Calculate the perimeter of the base of Kata’s task, in terms of a.
8. Show that the height, h, of Kata’s task can be written as $h = \frac{120}{a}$.
9. Find an expression for $V$, the volume of Kata’s task, in terms of a.
10. Find the value of a that maximizes the volume of Kata’s task.

Chili collects data and performs a t-test to determine whether Asian women in a low variance of child differs from that of Europeans. The data is summarized in the table:

<table>
<thead>
<tr>
<th>Variance</th>
<th>Asian</th>
<th>European</th>
</tr>
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<tbody>
<tr>
<td>Low</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Medium</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>High</td>
<td>50</td>
<td>40</td>
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</table>

11. Write down the null hypothesis, $H_0$, for the test.
12. Write down the number of degrees of freedom.
13. Show that the squared value of t-statistic with a low variance of child differs from that of Europeans.

What’s on the website?

Visit www.oxfordsecondary.co.uk/ibmathsl for free access to the full worked solutions to each and every question in the Course Companion.

www.oxfordsecondary.co.uk/ibmathsl also offers you a range of GDC activities for the TI-Nspire to help support your understanding.

Alternative GDC instructions for all material in the book is given for the TI-84 Plus and Casio 9860GII calculators, so you can be sure you will be supported no matter what calculator you use.

Example 4

Solve the simultaneous linear equations:

\[
\begin{align*}
2x + 3y &= 11 \\
4x - y &= 7
\end{align*}
\]

Choose an equation in the ordinary way and read off the answer:

$x = 1$ and $y = 5$

You could also have $x = 3$ and $y = 1$.

Mathematics Standard Level

1. Write down the values of $p$ and $q$.

At $f(x) = 0$, the graph crosses the x-axis:

\[
f(x) = 2(x + p)(x - q)
\]

At $f(x) = 0$, either:

\[
x = -p \quad \text{or} \quad x = q = 0
\]

These points are at $(-1,0)$ and $(3,0)$.

$x = -p$, $(p = 1)$ or $x = q$, $(q = 3)$.

So $p = -1$ and $q = 3$. 

You could also have $p = 3$ and $q = 1$. 

Practice exam papers will help you to fully prepare for your examinations. Worked solutions can be found on the website www.oxfordsecondary.co.uk/ibmathsl.
Rational functions

CHAPTER OBJECTIVES:

2.5 The reciprocal function \( x \mapsto \frac{1}{x}, x \neq 0 \), its graph and self-inverse nature

The rational function \( x \mapsto \frac{ax + b}{cx + d} \) and its graph

Vertical and horizontal asymptotes

Applying rational functions to real-life situations

Before you start

You should know how to:

1. Expand polynomials.
   e.g. Multiply the polynomials
   a. \(-2(3x - 1)\)
   b. \(3x(x^2 + 1)\)
   a. \(-2(3x - 1) = -6x + 2\)
   b. \(3x(x^2 + 1) = 3x^3 + 3x\)

2. Graph horizontal and vertical lines.
   e.g. Graph the lines
   \(y = x, y = -x, x = 2, x = -1, y = 3, y = -2\) on the same graph.

3. Recognize and describe a transformation.
   e.g. Find the translations that map \( y = x^2 \) onto \( A \) and \( B \).
   \( A \) is a horizontal shift of 2 units to the right. Function \( A \) is \( y = (x - 2)^2 \).
   \( B \) is a vertical shift of 3 units up. Function \( B \) is \( y = x^2 + 3 \).

Skills check

1. Expand the polynomials.
   a. \(-4(2x - 5)\)
   b. \(6(2x - 3)\)
   c. \(-x(x^2 + 7)\)
   d. \(x^2(x + 3)^2\)
   e. \(x(x - 3)(x + 8)\)

2. Draw these lines on one graph.
   \(x = 0, y = 0, x = 3, x = -2, y = -3, y = 4\)

3. Describe the transformations that map \( y = x^3 \) onto functions \( A \) and \( B \) and write down the equations of \( A \) and \( B \).
If you have an MP3 player, do you know how many songs, albums, sounds and so on can you fit on it? The answer depends on the quality of the recording setting and the length of the song. However, a rough idea is that a 4GB MP3 player will hold 136 hours or 8160 minutes of music. That’s approximately

- 2000 songs of 4 minutes
- or 1000 songs of 8 minutes
- or 4000 songs of 2 minutes.

This leads us to the function \( s = \frac{8000}{m} \) where \( s \) is the number of songs and \( m \) is the number of minutes that a song lasts.

This function is an example of the reciprocal function, \( f(x) = \frac{k}{x} \). In this chapter, you will use a GDC to explore the graphs of reciprocal functions and other rational functions that can be expressed in the form \( f(x) = \frac{ax + b}{cx + d} \). You will examine horizontal and vertical asymptotes for the graphs of these functions and the domain and ranges of the functions.
5.1 Reciprocals

Investigation – graphing product pairs

Think of pairs of numbers whose product is 24.
E.g. 24 × 1, 12 × 2, 8 × 3, 3 × 8. Copy the table and add some more pairs of numbers.

<table>
<thead>
<tr>
<th>x</th>
<th>24</th>
<th>12</th>
<th>8</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

Show your pairs as coordinates on a graph with 0 ≤ x ≤ 24 and 0 ≤ y ≤ 24.
Now try the same idea with negatives, e.g. −12 × −2 and graph these too.
Explain what you notice about
• the value of x as y gets bigger
• the value of y as x gets bigger
• the end behavior of your graph.

➔ The reciprocal of a number is 1 divided by that number.

For example, the reciprocal of 2 is \( \frac{1}{2} \)
Taking the reciprocal of a fraction turns it upside down.
For example, the reciprocal of \( \frac{3}{4} \) is \( \frac{4}{3} = 1 \times \frac{4}{3} = \frac{4}{3} \)
The reciprocal of \( \frac{7}{10} \) is \( \frac{10}{7} \). The reciprocal of \( \frac{1}{4} \) is \( \frac{4}{1} \) or 4.

➔ A number multiplied by its reciprocal equals 1.
For example \( 3 \times \frac{1}{3} = 1 \)

Example 1

Find the reciprocal of \( 2 \frac{1}{2} \)

Answer

\[
2 \frac{1}{2} = \frac{5}{2} \\
\text{Reciprocal of } \frac{5}{2} = \frac{2}{5}
\]

Write as an improper fraction. Turn it upside down.

You can find reciprocals of algebraic terms too.

➔ The reciprocal of \( x \) is \( \frac{1}{x} \) or \( x^{-1} \) and \( x^{-1} \times x = 1 \)

End behavior is the appearance of a graph as it is followed further and further in either direction.

Zero does not have a reciprocal as \( \frac{1}{0} \) is undefined. What does your GDC show for \( 1 \div 0? \)

Geometrical quantities in inverse proportion were described as reciprocals in a 1570 translation of Euclid’s Elements from 300BCE.

Check: \( \frac{5}{2} \times \frac{2}{5} = 1 \)

The reciprocal of a number or a variable is also called its multiplicative inverse.
**Exercise 5A**

1. Find the reciprocals.
   - \(a\) \(\frac{2}{3}\)  
   - \(b\) \(\frac{3}{2}\)  
   - \(c\) \(-3\)  
   - \(d\) \(-1\)  
   - \(e\) \(\frac{3}{2}\)  
   - \(f\) \(\frac{7}{11}\)  
   - \(g\) \(-\frac{3}{2}\)  
   - \(h\) \(\frac{12}{11}\)  

2. Find the reciprocals.
   - \(a\) \(6.5\)  
   - \(b\) \(x\)  
   - \(c\) \(y\)  
   - \(d\) \(3x\)  
   - \(e\) \(4y\)  
   - \(f\) \(\frac{2x}{9}\)  
   - \(g\) \(\frac{3a}{5}\)  
   - \(h\) \(\frac{2}{3d}\)  
   - \(i\) \(\frac{d}{t}\)  
   - \(j\) \(\frac{x+1}{x-1}\)  

3. Multiply each quantity by its reciprocal. Show your working.
   - \(a\) \(6\)  
   - \(b\) \(\frac{3}{4}\)  
   - \(c\) \(\frac{2c}{3d}\)  

4. a. What is the reciprocal of the reciprocal of 4?
   b. What is the reciprocal of the reciprocal of \(x\)?

5. For the function \(xy = 24\)
   - a. Find \(y\) when \(x\) is i. \(48\) ii. \(480\) iii. \(4800\) iv. \(48000\)
   b. What happens to the value of \(y\) when \(x\) gets larger?
   c. Will \(y\) ever reach zero? Explain.
   d. Find \(x\) when \(y\) is i. \(48\) ii. \(480\) iii. \(4800\) iv. \(48000\)
   e. What happens to the value of \(x\) when \(y\) gets larger?
   f. Will \(x\) ever reach zero? Explain.

---

**5.2 The reciprocal function**

The reciprocal function is

\[ f(x) = \frac{k}{x} \]

where \(k\) is a constant.

Graphs of reciprocal functions all have similar shapes.

---

**Investigation – graphs of reciprocal functions**

Use your GDC to draw all the graphs in this investigation.

1. Draw a graph of \(a\) \(f(x) = \frac{1}{x}\) \(b\) \(g(x) = \frac{2}{x}\) \(c\) \(h(x) = \frac{3}{x}\)
   What is the effect of changing the value of the numerator?

2. Draw a graph of \(a\) \(f(x) = \frac{-1}{x}\) \(b\) \(g(x) = \frac{-2}{x}\) \(c\) \(h(x) = \frac{-3}{x}\)
   What is the effect of changing the sign of the numerator?

3. a. Copy and complete this table for \(f(x) = \frac{4}{x}\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>0.25</th>
<th>0.4</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>10</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. What do you notice about the values of \(x\) and \(f(x)\) in the table?
   c. Draw the graph of the function.
   d. Draw the line \(y = x\) on the same graph.
   e. Reflect \(f(x) = \frac{4}{x}\) in the line \(y = x\).
   f. What do you notice?
   g. What does this tell you about the inverse function \(f^{-1}\)?
Asymptotes

The graphs of the functions \( f(x) \), \( g(x) \) and \( h(x) \) in the Investigation on page 143 all consist of two curves. The curves get closer and closer to the axes but never actually touch or cross them.

The axes are asymptotes to the graph.

➔ If a curve gets continually closer to a straight line but never meets it, the straight line is called an asymptote.

\( y = b \) is an asymptote to the function \( y = f(x) \)

As \( x \to \infty \), \( f(x) \to b \)

The symbol \( \to \) means ‘approaches’.

➔ The graph of any reciprocal function of the form \( y = \frac{k}{x} \) has a vertical asymptote \( x = 0 \) and a horizontal asymptote \( y = 0 \)

The graph of a reciprocal function is called a hyperbola.

➔ The \( x \)-axis is the horizontal asymptote.

➔ The \( y \)-axis is the vertical asymptote.

➔ Both the domain and range are all the real numbers except zero.

➔ The two separate parts of the graph are reflections of each other in \( y = -x \)

➔ \( y = -x \) and \( y = x \) are lines of symmetry for this function.

In Chapter 1 you saw that to draw the inverse function of \( f(x) \), you reflect its graph in the line \( y = x \). If you reflect \( f(x) = \frac{1}{x} \) in the line \( y = x \) you get the same graph as for \( f(x) \).

➔ The reciprocal function is a self-inverse function

The equation of the function in the Investigation on page 142 is \( xy = 24 \). It can be written as \( y = \frac{24}{x} \) and is a reciprocal function. It has a graph similar to the one shown above.
The design of the Yas Hotel in Abu Dhabi (designed by Asymptote Architecture) is based on mathematical models. It also has a Formula 1 racetrack running through the centre of the hotel!

**Example 2**

For each function:
- write down the equations of the vertical and horizontal asymptotes
- sketch the graph
- state the domain and range.

\[ a \quad y = \frac{9}{x} \quad b \quad y = \frac{9}{x} + 2 \]

**Answers**

**a**  Asymptotes are \( x = 0 \) and \( y = 0 \)

The graph of \( f(x) + 2 \) is the same as the graph of \( f(x) \) but shifted 2 units in the y-direction.

**b**  Asymptotes are \( x = 0 \) and \( y = 2 \)

Domain \( x \in \mathbb{R}, x \neq 0 \),
range \( y \in \mathbb{R}, y \neq 0 \)
Exercise 5B

1. Draw these on separate graphs.
   a) \( y = \frac{5}{x} \)
   b) \( y = \frac{6}{x} \)
   c) \( xy = 8 \)

2. On the same graph show \( y = \frac{12}{x} \) and \( y = \frac{-12}{x} \).

3. a) Sketch the graph of \( f(x) = \frac{1}{x} \) and write down its asymptotes.
    b) Sketch the graph of \( f(x) = \frac{1}{x} + 2 \) and write down its asymptotes.

4. Identify the horizontal and vertical asymptotes of these functions and then state their domain and range.
   a) \( y = \frac{20}{x} \)
   b) \( y = \frac{3}{x} + 2 \)
   c) \( y = \frac{4}{x} - 2 \)

5. The Corryvreckan, the third largest whirlpool in the world, is between the islands of Jura and Scarba off the coast of Scotland. Flood tides and inflow from the west and the roar of the resulting maelstrom can be heard 16 km away.

   The speed of the surrounding water increases as you approach the center and is modeled by \( s = \frac{250}{d} \) where \( s \) is the speed of the water in m/s\(^{-1} \) and \( d \) is the distance from the center in metres.
   a) Use your GDC to sketch the function with \( 0 \leq d \leq 50 \) and \( 0 \leq s \leq 200 \).
   b) At what distance is the speed 10 m/s\(^{-1} \)?
   c) What is the speed of the water 100 m from the center?

6. The force (\( F \)) required to raise an object of mass 1500 kg is modeled by \( F = \frac{1500}{l} \) where \( l \) is the length of the lever in metres and the force is measured in newtons.
   a) Sketch the graph with \( 0 \leq l \leq 6 \) and \( 0 \leq F \leq 5000 \).
   b) How much force would you need to apply if you had a 2 m lever?
   c) How long would the lever need to be if you could manage a force of
      i) 1000 N
      ii) 2000 N
      iii) 3000 N?
5.3 Rational functions

Have you noticed the way the sound of a siren changes as a fire engine or police car passes you? The observed frequency is higher than the emitted frequency during the approach, it is identical at the instant of passing by, and it is lower during the time it moves away. This is called the Doppler effect. The equation for the observed frequency of sound when the source is traveling toward you is:

\[
 f_i = \frac{330f}{330 - v}
\]

where

- 330 is the speed of sound in m s\(^{-1}\)
- \(f_i\) is the observed frequency in Hz
- \(f\) is the emitted frequency
- \(v\) is the velocity of the source toward you

\(f_i\) is a rational function.

A rational function is a function of the form \(f(x) = \frac{g(x)}{h(x)}\) where \(g\) and \(h\) are polynomials.

In this course \(g(x)\) and \(h(x)\) will be restricted to linear functions of the form \(px + q\) so we can investigate rational functions \(f(x)\) where

\[
 f(x) = \frac{ax + b}{cx + d}
\]

Example 3

A vehicle is coming towards you at 96 km h\(^{-1}\) (60 miles per hour) and sounds its horn with a frequency of 8000 Hz. What is the frequency of the sound you hear if the speed of sound is 330 m s\(^{-1}\)?

Answer

96 km h\(^{-1}\) = 96000 m h\(^{-1}\)
96000 m h\(^{-1}\) = \(\frac{96000}{3600}\) = 26.7 m s\(^{-1}\)
Observed frequency = \(\frac{330f}{330 - v}\)
= \(\frac{330 \times 8000}{330 - 26.7}\)
= 8700 Hz (3sf)

Sound frequency is measured in hertz (Hz), the number of waves per second.

A rational function is a function of the form \(f(x) = \frac{g(x)}{h(x)}\) where \(g\) and \(h\) are polynomials.

\(h(x)\) cannot be zero since a value divided by zero is undefined.

The units of speed must all be the same in the equation. You can round numbers to get an approximate answer.
Investigation – graphing rational functions 1

a Use your GDC to show sketches of \( y = \frac{1}{x}, \ y = \frac{1}{x-2}, \ y = \frac{1}{x+3} \) and \( y = \frac{2}{x+3} \).

b Copy and complete the table.

<table>
<thead>
<tr>
<th>Rational function</th>
<th>Vertical asymptote</th>
<th>Horizontal asymptote</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{1}{x+3} )</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{2}{x+3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c What effect does changing the denominator have on the vertical asymptote?

d What do you notice about the horizontal asymptotes?

e What do you notice about the domain and the value of the vertical asymptote?

f What do you notice about the range and the value of the horizontal asymptote?

Rational functions of the form \( y = \frac{k}{x-b} \)

A rational function \( y = \frac{k}{x-b} \), where \( k \) and \( b \) are constants, will have a vertical asymptote when the denominator equals zero, that is, when \( x = b \).

The horizontal asymptote will be the \( x \)-axis.

Example 4

a Identify the horizontal and vertical asymptotes of \( y = \frac{1}{x-3} \).

b State the domain and range.

c Sketch the function with the help of your GDC.

Answers

a The \( x \)-axis (\( y = 0 \)) is the horizontal asymptote. \( x = 3 \) is the vertical asymptote.

\( \frac{1}{0} \) is undefined. We will consider this in more detail in the Theory of Knowledge section at the end of the chapter.

b Since the numerator will never be 0, the graph of this function never touches the \( x \)-axis.

The denominator is zero when \( x = 3 \).

You may wish to explore the concept of infinity.

Continued on next page
Exercise 5C

1 Identify the horizontal and vertical asymptotes of these functions and state their domain and range.
   a \( y = \frac{1}{x+1} \)  b \( y = \frac{1}{x-4} \)  c \( y = \frac{-2}{x-5} \)  d \( y = \frac{4}{x+1} \)
   e \( y = \frac{4}{x+1} + 2 \)  f \( y = \frac{4}{x+1} - 2 \)  g \( y = \frac{4}{x-3} + 2 \)  h \( y = \frac{-2}{x+3} - 2 \)

2 Sketch each function with the help of your GDC and state the domain and range.
   a \( y = \frac{4}{x} \)  b \( y = \frac{3}{x-3} + 1 \)  c \( y = \frac{-4}{x+5} - 8 \)
   d \( y = \frac{1}{x-7} + 3 \)  e \( y = \frac{6}{x+2} - 6 \)  f \( y = \frac{5}{x} + 4 \)
   g \( y = \frac{1}{4x+12} - 2 \)  h \( y = \frac{3}{2x} \)  i \( y = \frac{4}{3x-6} + 5 \)

3 When lightning strikes, the light reaches your eyes virtually instantaneously. But the sound of the thunder travels at approximately 331 m s\(^{-1}\). However, sound waves are affected by the temperature of the surrounding air. The time sound takes to travel one kilometre is modeled by 
   \[ t = \frac{1000}{0.6c + 331} \]
   where \( t \) is the time in seconds and \( c \) is the temperature in degrees Celsius.
   a Sketch the graph of \( t \) for temperatures from \(-20^\circ C\) to \(40^\circ C\).
   b If you are one kilometre away and it is 3 seconds before you hear the thunder, what is the temperature of the surrounding air?

4 a On the same set of axes, sketch \( y = x + 2 \) and \( y = \frac{1}{x+2} \)
   Compare the two graphs and make connections between the linear function and its reciprocal function.
   b Now do the same for \( y = x + 1 \) and \( y = \frac{1}{x+1} \)
Rational functions of the form \(y = \frac{ax + b}{cx + d}\)

Every rational function of the form \(y = \frac{ax + b}{cx + d}\) has a graph called a hyperbola.

The graph of any rational function \(y = \frac{ax + b}{cx + d}\) has a vertical and a horizontal asymptote.

**Investigation – graphing rational functions 2**

a Use your GDC to show sketches of
\[y = \frac{x}{x + 3}, \quad y = \frac{x + 1}{x + 3}, \quad y = \frac{2x}{x + 3} \quad \text{and} \quad y = \frac{2x - 1}{x + 3}\]

b Copy and complete the table.

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<td>(y = 0)</td>
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<td>(y = 0)</td>
<td>(x \neq -3)</td>
<td>(y \in \mathbb{R})</td>
</tr>
</tbody>
</table>

c What do you notice about the horizontal asymptotes?
d What do you notice about the domain and the value of the vertical asymptote?

→ The vertical asymptote occurs at the \(x\)-value that makes the denominator zero.
→ The horizontal asymptote is the line \(y = \frac{a}{c}\)

To find the horizontal asymptote rearrange the equation to make \(x\) the subject.

\[
y = \frac{ax + b}{cx + d}
\]

\[
y(cx + d) = ax + b
\]

\[
cyx - ax = b - dy
\]

\[
x = \frac{b - dy}{cy - a}
\]

The horizontal asymptote occurs when the denominator is zero, that is, when

\(cy = a\) or \(y = \frac{a}{c}\)
Example 5

For the function \( y = \frac{x + 1}{2x - 4} \)

a sketch the graph

b find the vertical and horizontal asymptotes

c state the domain and range.

Answers

a

\[
\begin{array}{c|c}
\text{ } & y \\
\hline
-8 & 6 \\
-6 & 4 \\
-4 & 2 \\
-2 & 1 \\
0 & y \\
2 & -1 \\
4 & -3 \\
6 & -4 \\
8 & -6 \\
\end{array}
\]

Vertical asymptote \( x = 2 \)

Horizontal asymptote \( y = \frac{1}{2} \)

c Domain \( x \in \mathbb{R}, x \neq 2 \)

Range \( y \in \mathbb{R}, y \neq \frac{1}{2} \)

Exercise 5D

1 Identify the horizontal and vertical asymptotes of these functions and then state the domain and range.

\begin{align*}
a & \quad y = \frac{x + 2}{x - 3} \\
b & \quad y = \frac{2x + 2}{3x - 1} \\
c & \quad y = \frac{-3x + 2}{-4x - 5} \\
d & \quad y = \frac{34x - 2}{16x + 4}
\end{align*}

2 Match the function with the graph.

\begin{align*}
a & \quad y = \frac{5}{x} \\
b & \quad y = \frac{x + 2}{x - 2} \\
c & \quad y = \frac{x - 1}{x - 3} \\
d & \quad y = \frac{1}{x - 4}
\end{align*}
3 Sketch each function using your GDC and state the domain and range.

\[ a \quad y = \frac{x + 2}{x + 3} \]
\[ b \quad y = \frac{x}{4x + 3} \]
\[ c \quad y = \frac{x - 7}{3x - 8} \]
\[ d \quad y = \frac{9x + 1}{3x - 2} \]
\[ e \quad y = \frac{-3x + 10}{4x - 12} \]
\[ f \quad y = \frac{5x + 2}{4x} \]
\[ g \quad y = \frac{3x}{2x - 4} \]
\[ h \quad y = \frac{7x}{x - 15} \]
\[ i \quad y = \frac{14x - 4}{2x - 1} \]

4 Write a rational function that has a vertical asymptote at \( x = -4 \) and a horizontal asymptote at \( y = 3 \).

5 Chris and Lee design T-shirts for surfers and set up a T-shirt printing business in their garage. It will cost $450 to set up the equipment and they estimate that it will cost $5.50 to print each T-shirt.

a Write a linear function \( C(x) \) giving the total cost of producing \( x \) T-shirts. Remember to take the set-up cost into account.

b Write a rational function \( A(x) \) giving the average cost per T-shirt of producing \( x \) of them.

c What is the domain of \( A(x) \) in the context of the problem? Explain.

d Write down the vertical asymptote of \( A(x) \).

e Find the horizontal asymptote for \( A(x) \). What meaning does this value have in the context of the problem?

EXAM-STYLE QUESTION

6 Young’s rule is a way of calculating doses of medicine for children over the age of two, based on the adult dose.

‘Take the age of the child in years and divide by their age plus 12. Multiply this number by the adult dose.’

This is modeled by the function \( c = \frac{at}{t + 12} \) where \( c \) is the child’s dose, \( a \) is the adult dose in mg and \( t \) is the age of the child in years.
a Make a table of values for ages 2 to 12 with an adult dose of 100 mg.
b Use your values from a to draw a graph of the function.
c Use the graph to estimate the dose for a 7 1/2-year old.
d Write down the equation of the horizontal asymptote.
e What does the value of the horizontal asymptote mean for Young’s rule?

7 The average cost of electricity per year for a refrigerator is $92.
a A new refrigerator costs $550. Determine the total annual cost for a refrigerator that lasts for 15 years. Assume costs include purchase and electricity.
b Develop a function that gives the annual cost of a refrigerator as a function of the number of years you own the refrigerator.
c Sketch a graph of that function. What is an appropriate window? Label the scale.
d Since this is a rational function, determine its asymptotes.
e Explain the meaning of the horizontal asymptote in terms of the refrigerator.
f A company offers a refrigerator that costs $1200, but says that it will last at least twenty years. Is this refrigerator worth the difference in cost?

Review exercise

EXAM-STYLE QUESTION

1 Match the function with the graph.

\[
\begin{align*}
\text{i} & \quad f(x) = \frac{2}{x+2} \\
\text{ii} & \quad f(x) = \frac{1}{x-3} \\
\text{iii} & \quad f(x) = \frac{4x+1}{x} \\
\text{iv} & \quad f(x) = \frac{1-x}{x} \\
\text{v} & \quad f(x) = \frac{x-2}{x-4} \\
\text{vi} & \quad f(x) = \frac{x+2}{x+4}
\end{align*}
\]
2 Given \( f(x) = \frac{5}{x} \)  \( f(x) = \frac{1}{x+1} \)  \( f(x) = \frac{x+3}{3-x} \)

i Sketch the function.

ii Determine the vertical and horizontal asymptotes of the function.

iii Find the domain and range of the function.

3 For each of these functions, write down the asymptotes, domain and range.
4 A group of students want to give their teacher a voucher for a weekend at a health spa. The voucher costs $300.
   a If $c$ represents the cost for each student and $s$ represents the number of students, write an equation to show the cost in terms of the number of students.
   b Draw a graph of the function.
   c Explain any limitations on the range and domain of this function.

5 The function $f$ is given by
   \[ f(x) = \frac{2x - 1}{x + 2}, \quad x \in \mathbb{R}, \quad x \neq -2 \]
   a i Find the horizontal asymptote of the graph of $y = f(x)$
       ii Find the vertical asymptote of the graph.
       iii Write down the coordinates of the point $P$ at which the asymptotes intersect.
   b Find the points of intersection of the graph with the axes.
   c Hence sketch the graph of $y = f(x)$, showing the asymptotes by dotted lines.

**Review exercise**

**EXAM-STYLE QUESTION**

1 Sketch each function with the help of your GDC. State the domain and range.
   a $f(x) = -\frac{6}{x}$
   b $f(x) = \frac{2}{x} + 3$
   c $f(x) = \frac{-2}{x - 5}$
   d $f(x) = \frac{3}{x - 7} - 8$
   e $f(x) = \frac{8}{x + 3}$
   f $f(x) = \frac{-6}{x + 4} - 2$

2 An airline flies from London to New York, which is a distance of 5600 km.
   a Show that this information can be written as $s = \frac{5600}{t}$
      where $s$ is the average speed of the plane in km h$^{-1}$ and $t$ is the time in hours.
   b Sketch a graph of this function with $0 \leq s \leq 1200$ and $0 \leq t \leq 20$.
   c If the flight takes 10 hours, what is the average speed of the plane?
**Exam-Style Questions**

3. People with sensitive skin must be careful about the amount of time spent in direct sunlight. The relation

\[ m = \frac{22.2s + 1428}{s} \]

where \( m \) is the time in minutes and \( s \) is the sun scale value, gives the maximum amount of time that a person with sensitive skin can spend in direct sunlight without skin damage.

a. Sketch this relation when \( 0 \leq s \leq 120 \) and \( 0 \leq m \leq 300 \)

b. Find the number of minutes that skin can be exposed when
   i. \( s = 10 \)
   ii. \( s = 40 \)
   iii. \( s = 100 \)

c. What is the horizontal asymptote?

d. Explain what this represents for a person with sensitive skin.

4. The city mayor is giving out face masks during a flu outbreak in Bangkok. The cost \( (c) \) in Thai baht for giving masks to \( m \) percent of the population is given by

\[ c = \frac{750000m}{100 - m} \]

a. Choose a suitable scale and use your GDC to help sketch the function.

b. Find the cost of supplying
   i. 20%
   ii. 50%
   iii. 90%
   of the population.

b. Would it be possible to supply all of the population using this model? Explain your answer.

5. The function \( f(x) \) is defined as

\[ f(x) = 2 + \frac{1}{2x - 5}, \quad x \neq \frac{5}{2} \]

a. Sketch the curve of \( f \) for \(-3 \leq x \leq 5\), showing the asymptotes.

b. Using your sketch, write down
   i. the equation of each asymptote
   ii. the value of the \( x \)-intercept
   iii. the value of the \( y \)-intercept.
CHAPTER 5 SUMMARY
Reciprocals
● The **reciprocal** of a number is 1 divided by that number.
● A number multiplied by its reciprocal equals 1.
  
  For example $3 \times \frac{1}{3} = 1$

● The **reciprocal** of $x$ is $\frac{1}{x}$ or $x^{-1}$ and $x \times x^{-1} = 1$

The reciprocal function
● If a curve gets continually closer to a straight line but never meets it, the straight line is called an **asymptote**.
● The graph of any reciprocal function of the form $y = \frac{k}{x}$ has a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$

● The graph of a reciprocal function is called a **hyperbola**.
  ■ The x-axis is the horizontal asymptote.
  ■ The y-axis is the vertical asymptote.
  ■ Both the domain and range are all the real numbers except zero.
  ■ The two separate parts of the graph are reflections of each other in $y = -x$
  ■ $y = x$ and $y = -x$ are lines of symmetry for this function.

● The reciprocal function is a **self-inverse function**.

Rational functions
● A rational function is a function of the form $f(x) = \frac{g(x)}{h(x)}$ where $g$ and $h$ are polynomials.
● Every rational function of the form $y = \frac{ax + b}{cx + d}$ has a graph called a hyperbola.
● The vertical asymptote occurs at the $x$-value that makes the denominator zero.
● The horizontal asymptote is the line $y = \frac{a}{c}$


**Egyptian fractions**

The ancient Egyptians only used fractions with a numerator of 1, for example: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc.

This meant that instead of $\frac{3}{4}$ they wrote $\frac{1}{2} + \frac{1}{4}$. Their fractions were all in the form $\frac{1}{n}$ and are called **unit fractions**.

Numbers such as $\frac{2}{7}$ were represented as sums of unit fractions (e.g. $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$).

Also, the same fraction could not be used twice (so $\frac{2}{7} = \frac{1}{7} + \frac{1}{7}$ was not allowed).

For example, $\frac{5}{8}$ would be $\frac{1}{2} + \frac{1}{8}$.

**In algebra:**

$\frac{3}{4x} = \frac{1}{2x} + \frac{1}{4x}$

- Write each algebraic expression as an Egyptian fraction.

  - $\frac{4}{3x}$
  - $\frac{5}{4x}$
  - $\frac{7}{4x}$
  - $\frac{23}{24x}$

**Where do you think this could be useful?**

**What are the limitations of these fractions?**

**Is it possible to write every fraction as an Egyptian fraction? How do you know?**

- Write these as unit fractions.

  - $\frac{5}{6}$
  - $\frac{5}{8}$
  - $\frac{2}{5}$
  - $\frac{6}{7}$

- In an Inca quipu, the knots in the strings represent numbers.

- The Rhind Mathematical Papyrus dated 1650 BCE contains a table of Egyptian fractions copied from another papyrus 200 years older!
Is there a difference between zero and nothing?

More than 2000 years ago, Babylonian and Hindu cultures had systems for representing an absence of a number. In the ninth century CE, the Islamic mathematician and philosopher Muhammad al-Khwarizmi remarked that if, in a calculation, no number appears in the place of tens, a little circle should be used ‘to keep the rows’. The Arabs called this circle sifr (empty). The name sifr eventually became our word zero.

- Does this mean that zero was nothing?

- Who first used zero?
- What was used before that?
- Make a list of all of the subsets of \( \{0, 1, 2, 3\} \).
- Notice that one subset is \( \{0\} \) and another is \( \{\} \).
- Now try this. Solve the equation \( 9 + x = 3^2 \) and the equation \( 3x = 0 \).
- We have 1 CE and 1 BCE. What about a year zero?
- The ancient Greeks were not sure what to do with zero and they questioned how nothing could be something. Zeno’s paradoxes (a good topic to research) depend in some part on the tentative use of zero.
- How did the Mayan and Inca cultures understand zero?
- Where is zero in the decimal system? Is it positive or negative?
- What happens if you divide zero by anything?
- What happens if you divide anything by zero?
- What happens if you divide zero by zero?