The Poisson distribution can be used to (at least approximately) model a large number of natural and social phenomena. You might not expect the number of photons arriving at a cosmic ray observatory, the number of claims made to an insurance company, the number of earthquakes of a given intensity and the number of atoms decaying in a radioactive material to have much in common, but they are all examples of this distribution. The photo is of VERITAS – Very Energetic Radiation Telescope Array in Arizona – which is helping to shape our understanding of how subatomic particles like photons are accelerated to extremely high energy levels.

**Objectives**

After studying this chapter you should be able to:
- calculate probabilities for the distribution $\text{Po}(\mu)$;
- use the fact that if $X \sim \text{Po}(\mu)$ then the mean and variance of $X$ are each equal to $\mu$;
- understand the relevance of the Poisson distribution to the distribution of random events, and use the Poisson distribution as a model;

**Before you start**

You should know how to:

1. Use your calculator to work out values of exponential functions.
   
   Find the value of $e^{-2.5}$
   
   $e^{-2.5} = 0.0821$

2. Substitute values into more complex formulae
   
   Find the value of $p = \frac{e^{-2.5} \times 2.5^4}{4!}$

   $\frac{e^{-2.5} \times 2.5^4}{4!} = \frac{0.0821 \times 39.06}{24} = 0.133$

**Skills check:**

1. Find the value of:
   
   a) $e^{-3}$
   
   b) $e^{-2.1}$

2. Find the value of $p = \frac{e^{-3} \times 3^5}{5!}$
1.1 Introducing the Poisson distribution

Think about the following random variables:

- The number of dandelions in a square metre of a piece of open ground.
- The number of errors in a page of a typed manuscript.
- The number of cars passing a point on a motorway in a minute.
- The number of telephone calls received by a company switchboard in half an hour.
- The number of lightning strikes in an area over a year.
Do they have any features in common? Does any one of them stand out as being rather different?

The behaviour of five of these random variables follow the Poisson distribution.

Formally, the conditions are that
i) events occur at random
ii) events occur independently of one another
iii) the average rate of occurrences remains constant
iv) there is zero probability of simultaneous occurrences.

The Poisson distribution is defined as
\[ P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!} \text{ for } r = 0, 1, 2, 3, \ldots \]

You need to have a value for \( \lambda \) in order for this to make sense, so there is a family of Poisson distributions but there is only one parameter, \( \lambda \), which is the mean number of occurrences in the time period (or length, or area) being considered.

Like the Binomial, you can write the Poisson distribution as \( X \sim \text{Po}(\lambda) \).

Example 1

If \( X \sim \text{Po}(3) \) find \( P(X = 2) \).

\[ P(X = 2) = \frac{e^{-3} 3^2}{2!} = 0.224 \text{ (3 s.f.)} \]

Example 2

The number of cars passing a point on a road during a five minute period may be modelled by the Poisson distribution with parameter 4.

Find the probability that in a five minute period

i) 2 cars go past

\[ X \sim \text{Po}(4); \]

\[ P(X = 2) = \frac{e^{-4} 4^2}{2!} = 0.146525\ldots = 0.147 \text{ (3 s.f.)} \]

ii) fewer than 3 cars go past.

\[ P(X = 0) = \frac{e^{-4} 4^0}{0!} = 0.01831\ldots = 0.0183 \text{ (3 s.f.)} \]

\[ P(X = 1) = \frac{e^{-4} 4^1}{1!} = 0.07326\ldots = 0.0733 \text{ (3 s.f.)} \]

\[ P(X < 3) = 0.01831 \ldots + 0.07326\ldots + 0.146525\ldots = 0.238 \text{ (3 s.f.)} \]
Mathematical note: It is not immediately obvious from the mathematics you cover in this course that the form of the Poisson constitutes a probability distribution – remember in S1 chapter 5 this requires all probabilities to be non-negative (which they obviously all are here) but also that the sum of the probabilities is 1.

\[ P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!} \quad \text{for } r = 0, 1, 2, 3, \ldots \] is a probability distribution because

\[ \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots = e^{\lambda} \] is an example of an advanced topic in Pure Maths where functions like exponentials, logarithms and the trigonometric functions have (infinite) power series forms. Truncated forms of these infinite series are how electronic calculators obtain values of these functions.

Exercise 1.1

1. If \( X \sim \text{Po}(2) \) find
   
   i) \( P(X = 1) \)  
   ii) \( P(X = 2) \)  
   iii) \( P(X = 3) \)

2. If \( X \sim \text{Po}(1.8) \) find
   
   i) \( P(X = 0) \)  
   ii) \( P(X = 1) \)  
   iii) \( P(X = 2) \)

3. If \( X \sim \text{Po}(5.3) \) find
   
   i) \( P(X = 3) \)  
   ii) \( P(X = 5) \)  
   iii) \( P(X = 7) \)

4. If \( X \sim \text{Po}(0.4) \) find
   
   i) \( P(X = 0) \)  
   ii) \( P(X = 1) \)  
   iii) \( P(X = 2) \)

5. If \( X \sim \text{Po}(2.15) \) find
   
   i) \( P(X = 2) \)  
   ii) \( P(X = 4) \)  
   iii) \( P(X = 6) \)

6. If \( X \sim \text{Po}(3.2) \) find
   
   i) \( P(X = 2) \)  
   ii) \( P(X \leq 2) \)  
   iii) \( P(X \geq 2) \)

7. The number of telephone calls arriving at an office switchboard in a five minute period may be modelled by a Poisson distribution with parameter 3.2. Find the probability that in a five minute period
   
   a) exactly 2 calls are received;  
   b) more than 2 calls are received.

8. The number of accidents which occur on a particular stretch of road in a day may be modelled by a Poisson distribution with parameter 1.3. Find the probability that on a particular day
   
   a) exactly 2 accidents occur on that stretch of road;  
   b) fewer than 2 accidents occur.
1.2 The role of the parameter of the Poisson distribution

The mean number of events in an interval of time or space is proportional to the size of the interval.

Example 2 in Section 1.1 looked at the number of cars passing a point on a road during a five minute period. This may be modelled by the Poisson distribution with parameter 4.

In this case, the number of cars passing that point in a twenty minute period may be modelled by the Poisson distribution with parameter 16, and in a one minute period may be modelled by the Poisson distribution with parameter 0.8.

If the conditions for a Poisson distribution are satisfied in a given period, they are also satisfied for periods of different length.

Example 3
The number of accidents in a week on a stretch of road is known to follow a Poisson distribution with parameter 2.1.

Find the probability that:

a) in a given week there is 1 accident
b) in a two week period there are 2 accidents
c) there is 1 accident in each of two successive weeks.

a) In 1 week, the number of accidents follows a Po(2.1) distribution,
so the probability of 1 accident = \( \frac{e^{-2.1} \cdot 2.1^1}{1!} \) = 0.257 (3 s.f.)

b) In 2 weeks, the number of accidents follows a Po(4.2) distribution,
so the probability of 2 accidents = \( \frac{e^{-4.2} \cdot 4.2^2}{2!} \) = 0.132 (3 s.f.)

c) This can not be done directly as a Poisson since it says what has to happen in each of two time periods, but these are the outcomes considered in part a.

So the probability this happens in two successive weeks is \( \left( \frac{e^{-2.1} \cdot 2.1^1}{1!} \right)^2 \) = 0.0661 (3 s.f.)

If the average rate of occurrences remains constant, then the mean number of occurrences in an interval will be proportional to the length of the interval.

This is considerably less than the probability in part b, and this is because ‘1 accident in each of two successive weeks’ will give ‘2 accidents in a two week period’ but so will having none then two, or having two and then none, so the total probability that in a two week period there are 2 accidents must be higher than specifying there will be 1 in each week.
Example 4

The number of flaws in a metre length of dress material is known to follow a Poisson distribution with parameter 0.4. Find the probabilities that

a) there are no flaws in a 1 metre length
b) there is 1 flaw in a 3 metre length
c) there is 1 flaw in a piece of material which is half a metre long.

a) \( X \sim \text{Po}(0.4); \quad \Rightarrow \quad P(X = 0) = \frac{e^{-0.4} \times 0.4^0}{0!} = 0.670 \text{(3 s.f.)} \)

b) \( Y \sim \text{Po}(1.2); \quad \Rightarrow \quad P(Y = 1) = \frac{e^{-1.2} \times 1.2^1}{1!} = 0.361 \text{(3 s.f.)} \)

c) \( Z \sim \text{Po}(0.2); \quad \Rightarrow \quad P(Z = 1) = \frac{e^{-0.2} \times 0.2^1}{1!} = 0.164 \text{(3 s.f.)} \)

It is good practice to define new variable names when the interval changes. While all three of these calculations relate to the same basic situation, they all use different Poisson distributions and this is a simple way to stop confusion arising.

Exercise 1.2

1. The number of telephone calls arriving at an office switchboard in a five minute period may be modelled by a Poisson distribution with parameter 1.4. Find the probability that in a ten minute period
   a) exactly 2 calls are received;
   b) more than 2 calls are received.

2. The number of accidents which occur on a particular stretch of road in a day may be modelled by a Poisson distribution with parameter 0.4. Find the probability that during a week (7 days)
   a) exactly 2 accidents occur on that stretch of road;
   b) fewer than 2 accidents occur.

3. The number of letters delivered to a house on a day may be modelled by a Poisson distribution with parameter 0.8.
   a) Find the probability that there are 2 letters delivered on a particular day.
   b) The home owner is away for 3 days. Find the probability that there will be more than 2 letters waiting for him when he gets back.

4. The number of errors on a page of a booklet can be modelled by a Poisson distribution with parameter 0.2.
   a) Find the probability that there is exactly 1 error on a given page.
   b) A section of the booklet has 7 pages. Find the probability that there are no more than 2 errors in the section.
   c) The booklet has 25 pages altogether. Find the probability that the booklet contains exactly 6 errors altogether.
5. The number of people calling a car breakdown service can be modelled by a Poisson distribution, and the service has an average of 6 calls per hour. Find the probability that in a half hour period
   a) exactly 2 calls are received;
   b) more than 2 calls are received.

1.3 The recurrence relation for the Poisson distribution

This is not directly in the course, but it is a useful property of the Poisson to be aware of, and gives some insight into why the Poisson distribution has the shape that it does.

You can calculate probabilities for a Poisson distribution in sequence using a recurrence relation.

Example 5

If \( X \sim \text{Po}(\lambda) \)

a) write down the probability that
   i) \( X = 3 \) and ii) \( X = 4 \).

b) write \( P(X = 4) \) in terms of \( P(X = 3) \)

\[
\begin{align*}
\text{a) i)} & \quad \frac{e^{-\lambda} \times \lambda^4}{4!} \\
\text{ii)} & \quad \frac{e^{-\lambda} \times \lambda^3}{3!}
\end{align*}
\]

\[
\begin{align*}
\text{b) } & \quad \frac{e^{-\lambda} \times \lambda^4}{4!} = \left( \frac{e^{-\lambda} \times \lambda^3}{3!} \right) \times \frac{\lambda}{4} \\
& \quad \text{so } P(X = 4) = P(X = 3) \times \frac{\lambda}{4} \\
& \quad 4! = 4 \times 3 \times 2 \times 1 = 4 \times (3 \times 2 \times 1) = 4 \times 3!
\end{align*}
\]

The general relationship is \( P(X = k + 1) = \frac{\lambda}{k+1} \times P(X = k) \)

These graphs show the probability distributions for different values of \( \lambda \) and what effect this has on the shape of a particular Poisson distribution.
All Poisson variables have an outcome space which is all of the non-negative integers. However, when \( \lambda \) is relatively low, the probabilities tail off very quickly.

\[
\frac{1.2}{1} = 1.2; \quad \frac{1.2}{2} = 0.6; \quad \frac{1.2}{3} = 0.4; \quad \frac{1.2}{4} = 0.3......
\]

so the initial probability that \( X = 0 \) is multiplied by 1.2, then 0.6, then 0.4, 0.3 .... and so the mode is when \( X = 1 \).

Here \( \lambda \) is larger than in the previous graph, and the peak has moved across to the right. For values of \( X \) which are less than \( \lambda \) the probability is higher than the previous probability, but once \( x \) is > \( \lambda \) the probabilities start to decrease. More values of \( x \) have a noticeable probability, so the highest individual probability is not as large as it was in the previous graph and the distribution is more spread out.

What happens when \( \lambda \) is an integer?
Here \( P(X = 4) = P(X = 3) \times \frac{4}{4} = P(X = 3) \) and the distribution has two modes – at 3 and 4. Generally, the mode of the Poisson (\( \lambda \)) distribution is at the integer below \( \lambda \) when \( \lambda \) is not an integer and there are 2 modes (at \( \lambda \) and \( \lambda - 1 \)) when it is an integer.

\( \lambda < 1 \) is a special case.
Here even the first time the recurrence relation is used you are multiplying by < 1, so the mode will be 0 and the probability distribution is strictly decreasing for all values of \( x \).
The general forms for the probabilities of 0 and 1 for a Poisson distribution are

\[
P(X = 0) = \frac{e^{-\lambda} \times \lambda^0}{0!} = e^{-\lambda} \quad \text{and} \quad P(X = 1) = \frac{e^{-\lambda} \times \lambda^1}{1!} = \lambda e^{-\lambda}
\]

**Example 6**

\(X \sim \text{Po}(\lambda)\) and \(P(X = 6) = 2 \times P(X = 5)\). Find the value of \(\lambda\).

\[P(X = 6) = \frac{\lambda^6}{6} \times P(X = 5) \quad \Rightarrow \frac{\lambda^6}{6} = 2 \quad \Rightarrow \lambda = 12.\]

**Example 7**

\(X \sim \text{Po}(5.8)\). State the mode of \(X\).

Since 5.8 is not an integer, the mode is the integer below it i.e. the mode is 5.

**Exercise 1.3**

1. \(X \sim \text{Po}(2.5)\).
   a) Write down an expression for \(P(X = 4)\) in terms of \(P(X = 3)\).
   b) If \(P(X = 3) = 0.214\), calculate the value of your expression in part a.
   c) Calculate \(P(X = 4)\) directly and check it is the same as your answer to b.
   d) What is the mode of \(X\)?

2. \(X \sim \text{Po}(5)\)
   a) Write down an expression for \(P(X = 5)\) in terms of \(P(X = 4)\)
   b) Explain why \(X\) has two modes at 4 and 5.

3. \(X \sim \text{Po}(\lambda)\) and \(P(X = 4) = 1.2 \times P(X = 3)\).
   a) Find the value of \(\lambda\).
   b) What is the mode of \(X\)?
1.4 Mean and variance of the Poisson distribution

If $X \sim \text{Po}(\lambda)$, then $E(X) = \lambda; \quad \text{Var}(X) = \lambda \Rightarrow \text{st. dev } (\sigma) = \sqrt{\lambda}$.

A special property of the Poisson distribution is that the mean and variance are always equal.

**Example 8**

The number of calls arriving at a company's switchboard in a ten minute period can be modelled by a Poisson distribution with parameter 3.5.

Give the mean and variance of the number of calls which arrive in

i) ten minutes     ii) an hour     iii) five minutes

i) here $\lambda = 3.5$ so the mean and variance will both be 3.5

ii) here $\lambda = 21 (= 3.5 \times 6)$ so the mean and variance will both be 21

iii) here $\lambda = 1.75 (= 3.5 \div 2)$ so the mean and variance will both be 1.75

**Example 9**

A dual carriageway has one lane blocked off because of roadworks.

The number of cars passing a point in a road in a number of one minute intervals is summarised in the table.

<table>
<thead>
<tr>
<th>number of cars</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>25</td>
<td>30</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

a) Calculate the mean and variance of the number of cars passing in one minute intervals.

b) Is the Poisson likely to provide an adequate model for the distribution of the number of cars passing in one minute intervals?

a) $\sum f = 70, \quad \sum xf = 228 \quad \sum x^2 f = 836$, so $\bar{x} = \frac{228}{70} = 3.26$ (3 s.f.)

and $\text{Var}(X) = \frac{\sum x^2 f}{\sum f} - \bar{x}^2 = \frac{836}{70} - \left(\frac{228}{70}\right)^2 = 1.33$ (3 s.f.)

b) The mean and variance are not numerically close so it is unlikely the Poisson will be an adequate model (with only one lane open for traffic, overtaking cannot happen on this stretch of the road and the numbers of cars will be much more consistent than would happen in normal circumstances – hence the variance is much lower than would be expected if the Poisson model did apply).
Derivation of Mean and Variance of the Poisson distribution*

You must be able to use these results but are not required to be able to prove them – they are included here for completeness, and as a nice manipulation using the power series expression for the exponential function.

\[ X \sim \text{Po}(\lambda) \iff \Pr\{X = k\} = \frac{e^{-\lambda} \lambda^k}{k!} \]

\[ E(X) = \sum_{k=0}^{\infty} k \times \frac{e^{-\lambda} \lambda^k}{k!} = 0 \times \frac{e^{-\lambda} \lambda^0}{0!} + \sum_{k=1}^{\infty} k \times \frac{e^{-\lambda} \lambda^k}{k!} \]

\[ = \lambda \times \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} \text{ – cancelling } k, \text{ after discarding the zero case.} \]

\[ = \lambda \]

\[ E(X^2) = \sum_{k=0}^{\infty} k^2 \times \frac{e^{-\lambda} \lambda^k}{k!} = \lambda \times \sum_{k=1}^{\infty} k \times \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} = \lambda \times \sum_{k=1}^{\infty} (k - 1 + 1) \times \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} \]

\[ = \lambda^2 \times \sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^{k-2}}{(k-2)!} + \lambda \times \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} \]

\[ = \lambda^2 + \lambda \]

Then \( \text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda \)

**Exercise 1.4**

1. If \( X \sim \text{Po}(3.2) \) find  
   i) \( E(X) \)  
   ii) \( \text{Var}(X) \)

2. If \( X \sim \text{Po}(49) \) find the mean and standard deviation of \( X \).

3. \( X \sim \text{Po}(3.6) \)
   a) Find the mean and standard deviation of \( X \).
   b) Find \( P(X > \mu) \), where \( \mu = E(X) \).
   c) Find \( P(X > \mu + 2\sigma) \), where \( \sigma \) is the standard deviation of \( X \).
   d) Find \( P(X < \mu - 2\sigma) \).

4. \( X \) is the number of telephone calls arriving at an office switchboard in a ten minute period. \( X \) may be modelled by a Poisson distribution with parameter 6.
   a) Find the mean and standard deviation of \( X \).
   b) Find \( P(X > \mu) \), where \( \mu = E(X) \).
   c) Find \( P(X > \mu + 2\sigma) \), where \( \sigma \) is the standard deviation of \( X \).
   d) Find \( P(X < \mu - 2\sigma) \).

5. Compare your answers to part d) of Q3 and Q4.
1.5 Modelling with the Poisson distribution

The Poisson describes the number of occurrences in a fixed period of time or space if the events occur independently of one another, at random and at a constant average rate.

Standard examples of Poisson processes in real life include: radioactive emissions, traffic passing a fixed point, telephone calls or letters arriving, and accidents occurring.

Example 10

The maternity ward of a hospital wanted to work out how many births would be likely to happen during a night.

The hospital has 3000 deliveries each year, so if these happen randomly around the clock 1000 deliveries would occur between the hours of midnight and 8.00 a.m. This is the time when many staff are off duty and it is important to ensure that there will be enough people to cope with the workload on any particular night.

The average number of deliveries per night is \( \frac{1000}{365} \), which is 2.74.

From this average rate the probability of delivering 0, 1, 2, etc. babies each night can be calculated using the Poisson distribution. If \( X \) is a random variable representing the number of deliveries per night, some probabilities are:

\[
P(X = 0) = 2.74^0 \times e^{-2.74} \frac{0!}{0!} = 0.065
\]

\[
P(X = 1) = 2.74^1 \times e^{-2.74} \frac{1!}{1!} = 0.177
\]

\[
P(X = 2) = 2.74^2 \times e^{-2.74} \frac{2!}{2!} = 0.242
\]

\[
P(X = 3) = 2.74^3 \times e^{-2.74} \frac{3!}{3!} = 0.221
\]

i) On how many days in the year would 5 or more deliveries be likely to occur?

ii) Over the course of one year, what is the greatest number of deliveries likely to occur at least once?

iii) Why might the pattern of deliveries not follow a Poisson distribution?

i) \( 52 = 365 \times P(X \geq 5) \)

ii) \( 8 \) – the largest value for which the probability is greater than \( \frac{1}{365} \)

iii) If deliveries were not random throughout the 24 hours.

   e.g. If a lot of women had labour induced or had elective caesareans done during the day.

Did you know?

An elective caesarean is planned in advance for some births which are expected to be difficult.
In this real life example, deliveries in fact followed the Poisson distribution very closely, and the hospital was able to predict the workload accurately.

The conditions for the Poisson distribution are that

i) events occur at random

ii) events occur independently of one another

iii) the average rate of occurrences remains constant

iv) there is zero probability of simultaneous occurrences.

Be careful:
Some change in the underlying conditions may alter the nature of the distribution, e.g. Traffic observed close to a junction, or where there are lane restrictions and traffic is funnelled into a queue travelling at constant speed.
The underlying conditions may be distorted by interference from other effects, e.g. if a birthday or Christmas occurs during the period considered then the Poisson conditions would not be reasonable for the arrival of letters for instance.

Randomness or independence may be lost due to a difference in the average rate of occurrences, e.g. the rate of traffic accidents occurring would be expected to vary somewhat as road conditions vary.

Example 11
The number of cyclists passing a remote village post-office during the day can be modelled as a Poisson random variable. On average two cyclists pass by in an hour.

a) What is the probability that
   i) no cyclist passes
   ii) more than 3 cyclists pass by between 10.00 and 11.00 am?

b) What is the probability that exactly one passes by while the shop-keeper is on a 20 minute tea-break?

c) What is the probability that more than 3 cyclists pass by in an hour exactly once in a six hour period?
a) In an hour (parts i and ii) $\lambda = 2$.
   
   i) $P(X = 0) = 0.1353$
   
   ii) $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.8571 = 0.1429$

b) In a 20 minute period $= \left( \frac{1}{3} \right)$ of an hour, the mean number of cyclists will be $2 \times \frac{1}{3} = \frac{2}{3}$

   \[ P(\text{exactly one}) = \frac{e^{-\frac{2}{3}} \left( \frac{2}{3} \right)^1}{1!} = 0.342 \text{ (3 s.f.)}. \]

b) Example 12

   At a certain harbour the number of boats arriving in a fifteen minute period can be modelled by a Poisson distribution with parameter 1.5

   a) Find the probability that exactly six boats will arrive in a period of an hour.

   b) Given that exactly six boats arrive in a period of an hour, find the conditional probability that there are twice as many arrive in the second half hour as arrive in the first half hour.

   a) in an hour the average number of boats arriving is 6, so

   \[ P(6 \text{ boats arrive in an hour}) = \frac{e^{-6} 6^6}{6!} = 0.161 \]

   b) if twice as many arrive in the second half hour, then there needs to be 2 in a half hour period and then 4 in the next half hour, so

   \[ P(2 \text{ boats arrive in half hour, then 4 boats in next half hour}) = \frac{e^{-3} 3^2}{2!} \times \frac{e^{-3} 3^4}{4!} = 0.224 \times 0.168 = 0.0376 \]

   Then the conditional probability is

   \[ P(2 \text{ then 4 in half hour} \mid 6 \text{ boats arrive in an hour}) = \frac{0.0376}{0.161} = 0.234 \]
Exercise 1.5

1. For the following random variables state whether they can be modelled by a Poisson distribution.
   If they can, give the value of the parameter \( \lambda \), if they cannot then explain why.
   
   a) The average number of cars per minute passing a point on a road is 12.  
      The traffic is flowing freely.  
      \( X \) = number of cars which pass in a 15 second period.
   
   b) The average number of cars per minute passing a point on a road is 14.  
      There are roadworks blocking one lane of the road.  
      \( X \) = number of cars which pass in a 30 second period.
   
   c) Amelie normally gets letters at an average rate of 1.5 per day.  
      \( X \) = number of letters Amelie gets on December 22nd.
   
   d) A petrol station which stays open all the time gets an average of 832 customers in a 24 hour time period.  
      \( X \) = number of customers in a quarter of an hour at the petrol station.
   
   e) An A&E department in a hospital treats 32 patients an hour on average.  
      \( X \) = number of patients treated between 5 pm and 7 pm on a Friday evening.

2. For the following situations state what assumptions are needed if a Poisson distribution is to be used to model them, and give the value of \( \lambda \) that would be used.
   You are **not** expected to do any calculations!
   
   a) On average defects in a roll of cloth occur at a rate of 0.2 per metre.  
      How many defects are there in a roll which is 8 m long?
   
   b) On average defects in a roll of cloth occur once in 2 metres.  
      How many defects are there in a roll which is 8 m long?
   
   c) A small shop averages 8 customers per hour.  
      How many customers does it have in twenty minutes?

3. An explorer thinks that the number of mosquito bites he gets when he is in the jungle will follow a Poisson distribution.
   The explorer records the number of mosquito bites he gets in the jungle during a number of hour long periods, and the results are summarised in the table:

<table>
<thead>
<tr>
<th>number of bites</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>≥7</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>3</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
a) Calculate the mean and variance of the number of bites the explorer gets in an hour in the jungle.

b) Do you think the Poisson is a good model for the number of bites the explorer gets in an hour in the jungle?

4. The number of emails Serena gets can be modelled by a Poisson distribution with a mean rate of 1.5 per hour.
   a) i) What is the probability that Serena gets no emails between 4 pm and 5 pm?
      ii) What is the probability that Serena gets more than 2 emails between 4 pm and 5 pm?
      iii) What is the probability that Serena gets one email between 6 pm and 6.20 pm?
   b) What is the probability that Serena gets more than 2 emails in an hour exactly twice in a five hour period?
   c) Would it be sensible to use the Poisson distribution to find the probability that Serena gets no emails between 4 am and 5 am?

5. The number of lightning strikes in the neighbourhood of a campsite in a week can be modelled by a Poisson distribution with parameter 1.5
   a) Find the probability that there is exactly one lightning strike in the neighbourhood in a given week.
   b) Alejandra spends three weeks at the campsite. Find the probability that there are exactly three lightning strikes in the neighbourhood during her holiday.
   c) Given that the neighbourhood has exactly three lightning strikes during her holiday, find the conditional probability that each week has exactly one strike.

Summary exercise 1

1. If $X \sim \text{P}(1.75)$ find
   a) $P(X = 2)$
   b) $P(X \leq 2)$
   c) $P(X \geq 2)$

2. If $X \sim \text{Po}(3.2)$
   a) find i) $P(X = 0)$  ii) $P(X = 1)$ iii) $P(X > 2)$
   b) For $X$, write down the i) mean ii) variance iii) standard deviation
c) Explain why the mode of $X$ is 3.

3. If $X \sim \text{Po}(6.4)$
   a) find i) $P(X \leq 3)$  ii) $P(X = 6)$
   b) For $X$, write down the i) mean ii) variance iii) standard deviation
c) Write down the mode of $X$.

   d) Find
      i) $P(X < \mu)$  ii) $P(|X - \mu| < \sigma)$
      where $\mu$ is the mean and $\sigma$ is the standard deviation of $X$. 

3. If $X \sim \text{Po}(6.4)$
   a) find i) $P(X \leq 3)$  ii) $P(X = 6)$
   b) For $X$, write down the i) mean ii) variance iii) standard deviation
c) Write down the mode of $X$.

   d) Find i) $P(X < \mu - \sigma)$ ii) $P(|X - \mu| < \sigma)$ where $\mu$ is the mean and $\sigma$ is the standard deviation of $X$. 

The Poisson distribution
4. The number of telephone calls arriving at Sharma’s home in a fifteen minute period may be modelled by a Poisson distribution with parameter 0.4. Find the probability that in an hour
a) exactly 2 calls are received
b) more than 2 calls are received.

5. The number of accidents which occur on a particular stretch of road in a day may be modelled by a Poisson distribution with parameter 1.6. Find the probability that on a particular day
a) exactly 2 accidents occur on that stretch of road.
b) less than 2 accidents occur.

6. X is the number of telephone calls arriving at an office switchboard in a ten minute period. X may be modelled by a Poisson distribution with parameter 4.
   a) Find the mean and standard deviation of X.
   b) Find \( P(X > \mu) \), where \( \mu = E(X) \).
   c) Find \( P(|X - \mu| > \sigma) \), where \( \sigma \) is the standard deviation of X.

7. An urban safety officer thinks that the number of traffic accidents in an area will follow a Poisson distribution.
The officer records the number of accidents in the area each week over a period of several months, and the results are summarised in the table.

<table>
<thead>
<tr>
<th>number of accidents</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>≥7</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

   a) Calculate the mean and variance of the number of accidents in the area in a week,
   b) Do you think the Poisson is a good model for the number of accidents in the area in a week?

8. The number of errors on a page of a book can be modelled by a Poisson distribution with parameter 0.15.
   a) Find the probability that there is exactly 1 error on a given page.
   b) A chapter of the book has 20 pages. Find the probability that there are no more than 2 errors in the chapter.
   c) What is the most likely number of errors in the chapter?

**Exam-style questions**

9. The number of errors on a page of the first proofs of a book can be modelled by a Poisson distribution with parameter 0.6.
   a) Find the probability that a page has exactly one error on it.
   b) Find the probability that a double page spread has exactly two errors on it.
   c) Given that a double page spread has exactly two errors on it, find the conditional probability that each page has exactly one error on it.

10. A shop sells spades. The demand for spades follows a Poisson distribution with mean 2.7 per week.
   a) Find the probability that the demand is exactly 2 spades in any one week.
   b) The shop has 4 spades in stock at the beginning of a week. Find the probability that this will be enough to satisfy the demand for spades in that week.
c) Given instead that there are \( n \) spades in stick, find, by trial and error, the least value of \( n \) for which the probability of not being able to satisfy the demand for spades in that week is less than 0.1.

11. The random variable \( X \) has the distribution \( \text{Po}(2.5) \). The random variable \( Y \) is defined by \( Y = 2X \).

a) Find the mean and variance of \( Y \).

b) Give a reason why the variable \( Y \) does not have a Poisson distribution.

12. People arrive randomly and independently at a supermarket checkout at an average rate of 2 people every 3 minutes.

a) Find the probability that exactly 4 people arrive in a 5-minute period. [2]

At another checkout in the same supermarket, people arrive randomly and independently at an average rate of 1 person each minute.

b) Find the probability that a total of fewer than 3 people arrive at the two checkouts in a 3-minute period. [3]

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13. The number of radioactive particles emitted per 150-minute period by some material has a Poisson distribution with mean 0.7.

a) Find the probability that at most 2 particles will be emitted during a randomly chosen 10-hour period. [3]

b) Find, in minutes, the longest time period for which the probability that no particles are emitted is at least 0.99. [5]

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Chapter summary

- The Poisson distribution is defined as
  \[
P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!} \quad \text{for } r = 0, 1, 2, 3, \ldots\]

- The Poisson distribution has a single parameter, \( \lambda \).

- The Poisson distribution is often written as \( X \sim \text{P}(\lambda) \).

- If \( X \sim \text{P}(\lambda) \), then \( E(X) = \lambda \); \( \text{Var}(X) = \sigma^2 = \lambda \Rightarrow \text{st. dev}(\sigma) = \sqrt{\lambda} \).

- The conditions for the Poisson are:
  i) events occur at random
  ii) events occur independently of one another
  iii) the average rate of occurrences remains constant
  iv) there is zero probability of simultaneous occurrences.