OXFORD MECHANICS for Cambridge International AS & A Level

Series editor: David Rayner
Martin Burgess, Phil Crossley, Jim Fensom
## Contents

### 1 Projectiles
- 1.1 Horizontal Projection  
- 1.2 Projection at an angle to the horizontal  
- 1.3 General formulae  

### 2 Moments
- 2.1 The unit of moment  
- 2.2 Forces at an angle  
- 2.3 Parallel forces and couples  
- 2.4 Non-parallel forces in equilibrium  

### 3 Centre of mass
- 3.1 Centres of mass of simple objects  
- 3.2 Centres of mass of common uniform bodies  
- Review exercise A  
- Maths in real-life: A risky business  

### 4 Equilibrium of rigid bodies (1)
- 4.1 The principle of equilibrium  

### 5 Hooke’s law
- 5.1 Elastic strings and springs  
- 5.2 Hooke’s law  
- 5.3 Elastic Potential Energy  

### 6 Circular Motion
- 6.1 Angular speed  
- 6.2 Strings supporting rotating particles  
- 6.3 Conical pendulums  
- 6.4 Rotating particles in contact with a surface  
- Review exercise B  
- Maths in real-life: Accelerating knowledge  

### 7 Equilibrium of rigid bodies (2)
- 7.1 Slipping and toppling  

### 8 Variable Forces
- 8.1 Acceleration as a function of time  
- Review exercise C  
- Maths in real-life: Modelling resistance  

### Answers  

### Index
The flight of a golf ball when struck into the air and the path of a football when kicked are both examples of projectile motion. Projectile motion is a form of motion in which an object (projectile) is given an initial velocity and then follows a path governed entirely by gravity. Thus, the projectile travels along a parabolic curve, accelerating downwards.

**Objectives**

- Model the motion of a projectile as a particle moving with constant acceleration and understand any limitations of the model.
- Use horizontal and vertical equations of motion to solve problems on the motion of projectiles, including finding the magnitude and direction of the velocity at a given time or position, the range on a horizontal plane and the greatest height reached.
- Derive and use the cartesian equations of the trajectory of a projectile, including problems in which the initial speed and/or the angle of projection may be unknown.

**Before you start**

You should know how to:

1. Use equations of motion with constant acceleration.
   
   **e.g.** Given that \( u = 2 \), \( a = -3 \), and \( t = 0.5 \), find \( v \) and \( s \).
   
   \[ v = u + at \Rightarrow 2 + (-3) \times 0.5 = 0.5 \]
   
   \[ v^2 = u^2 + 2as \Rightarrow 0.5^2 = 2^2 + 2 \times (-3) \times s \]
   
   \[ \Rightarrow s = 0.625 \]

**Skills check:**

1. a) Given that \( u = 16 \), \( a = -2 \), and \( t = 3 \), find \( v \) and \( s \).
   
   b) Given that \( s = 4 \), \( v = 10 \), and \( a = 3 \), find \( u \) and \( t \).
   
   c) Given that \( u = 1 \), \( v = -4 \), and \( t = 2 \), find \( a \) and \( s \).
2. Find the magnitude and direction of a vector.

   e.g. A vector has components as shown.

   Find the magnitude and direction of the vector.

   The magnitude of the vector is given by the length of $P$, and the direction is given by the angle $\theta$.

   $P^2 = 12^2 + 5^2$

   $P = 13$

   $\theta = \tan^{-1} \frac{5}{12} = 22.61986495 \approx 22.6^\circ$

When a particle is projected into the air it is called a projectile. We want to ask the question ‘what shape of path does a projectile follow?’ A ball thrown through the air might appear to follow a parabolic path, but can we prove this to be the case?

To simplify the situation some modeling assumptions need to be made. These assumptions are

1. That the projectile is modelled as a single particle
2. That air resistance is small enough to be negligible, and so the only force acting on the projectile is due to gravity.

### 1.1 Horizontal Projection

When considering projectile motion, we have to model the situation in two dimensions.

It is necessary to consider horizontal and vertical components of the projectile’s motion separately. We can apply the equations of motion under constant acceleration to each component of the particle’s motion.
We do this because the force of gravity which acts on the projectile only acts in the vertical plane. Thus, there is no force which acts in the direction of the horizontal component of motion, and so the horizontal velocity of the projectile remains constant.

The vertical component of motion is subject to the force of gravity and hence experiences a constant acceleration, \( g \). So the usual constant acceleration formulae apply in the vertical direction. As a reminder these are

\[
\begin{align*}
    v &= u + at \\
    v^2 &= u^2 + 2as \\
    s &= ut + \frac{1}{2} at^2 \\
    s &= \frac{1}{2} (u + v)t
\end{align*}
\]

Example 1

A particle is projected horizontally at 24 m s\(^{-1}\) from a point 125 m above a horizontal surface. Taking downwards as positive, find the time taken by the particle to reach the surface and the horizontal distance travelled in that time. Let \( g \approx 10 \) m s\(^{-2}\).

**Horizontal Projection**

\[
\begin{align*}
    u &= v = 24 \text{ m s}^{-1} \\
    s &= x \\
    t &= t \\
    a &= 0
\end{align*}
\]

Using \( s = ut + \frac{1}{2} at^2 \) for vertical motion gives

\[
125 = 0 + \frac{1}{2} \times 10 \times t^2
\]

\[
t^2 = \frac{2(125)}{10} = 25
\]

\[
t = 5 \text{ s}
\]

Using distance = velocity \times time for horizontal motion with \( t = 5 \) s gives

\[
\begin{align*}
    x &= 24 \times 5 \\
    x &= 120 \text{ m}
\end{align*}
\]

It is always helpful to draw a diagram to help you consider the horizontal and vertical components of motion separately.
Example 2

A particle is projected horizontally from a height with a speed of 12 ms\(^{-1}\). Taking downwards as positive, find the horizontal and vertical displacements of the particle from the point of projection after 2.5 seconds. Thus, find the distance of the particle from the point of projection after 2.5 seconds.

Horizontal motion
- \(u = v = 12\) ms\(^{-1}\)
- \(s = x\)
- \(t = 2.5\) s
- \(a = 0\)

Vertical motion
- \(u = 0\)
- \(v\) not required
- \(s = y\)
- \(t = 2.5\) s
- \(a = 10\) ms\(^{-2}\)

Using \(s = ut + \frac{1}{2}at^2\) for vertical motion gives

\[
y = 0 + \frac{1}{2} \times 10 \times 2.5^2
\]

\[
y = 31.25\) m
\]

Using \(s = ut\) for horizontal motion gives

- \(x = 12 \times 2.5\)
- \(x = 30\) m

Using Pythagoras’s Theorem, the distance \(d\) of particle from the point of projection is given by

\[
d = \sqrt{x^2 + y^2}
\]

\[
= \sqrt{30^2 + 31.25^2}
\]

\[
= 43.31930863
\]

\[
\approx 43.3\) m (3 s.f.)
\]
Example 3

A tennis ball is served horizontally with an initial speed of 23 ms\(^{-1}\) from a height of 2.8 m. The net is 1 m high and is situated 12 m horizontally from the server. Find the distance by which the ball clears the net.

Horizontal motion
\[ u = v = 23 \text{ ms}^{-1} \]
\[ s = 12 \text{ m} \]
\[ t = t \]
\[ a = 0 \]

Vertical motion
\[ u = 0 \]
\[ v \text{ not required} \]
\[ s = y \]
\[ t = t \]
\[ a = 10 \text{ ms}^{-2} \]

First consider horizontal motion to find the time it takes for the ball to pass over the net.
\[ s = ut \]
\[ 12 = 23t \]
\[ t = \frac{12}{23} \]

This is the time at which the ball is directly above the net.

Find the distance downwards (from initial height) that the ball has travelled when it passes over the net. Use \( s = ut + \frac{1}{2} at^2 \) for vertical motion.
\[ s = \frac{1}{2} \times 10 \times \left(\frac{12}{23}\right)^2 \]
\[ s = 1.36106 \approx 1.36 \text{ m} \text{ (3 s.f.)} \]

Thus, when the ball passes the net it is at a height of \( 2.8 - 1.36 \text{ m} = 1.44 \text{ m} \)

The ball clears the net by 0.44 m.
Example 4

A stone is thrown horizontally with an initial speed of 18 m/s from the top of a cliff 100 m above sea level. Find the speed and direction of the stone 1.5 s after it is thrown.

Horizontal motion
\( u = v = 18 \text{ m/s} \)
\( s = s \)
\( t = 1.5 \text{ s} \)
\( a = 0 \)

Vertical motion
\( u = 0 \)
\( v = v \)
\( s = s \) not required
\( t = 1.5 \text{ s} \)
\( a = 10 \text{ m/s}^2 \)

Using \( v = u + at \) for vertical motion gives
\( v = 0 + 10(1.5) = 15 \text{ m/s} \) downwards.

There is no acceleration in the direction of the horizontal component of motion, so the speed remains constant at 18 m/s.

Using Pythagoras’s Theorem:
\( V^2 = 18^2 + 15^2 \)
\( V = 23.43074903 \)
\( V \approx 23.4 \text{ m/s} \) (3 s.f.)

And using trigonometry with the horizontal and vertical velocities after 1.5 s
\( \tan \theta = \frac{15}{18} \)
\( \theta = 39.80557109 \)
\( \theta \approx 39.8^\circ \) (3 s.f.)

After 1.5 s, the stone has a speed of 23.4 m/s and is travelling in a direction of 39.8° below the horizontal.
Exercise 1.1

1. A particle is projected horizontally at 15 ms\(^{-1}\) from a point 80 m above a horizontal surface. Find
   a) the time taken for the particle to reach the surface
   b) the horizontal distance travelled in that time.

2. A stone is thrown horizontally at 21 ms\(^{-1}\) from the top of a vertical cliff. The height of the cliff is 35 m above sea level. Find the distance from the foot of the cliff to the point where the stone enters the sea.

3. A particle is projected horizontally from a point 5 m above a horizontal surface. The particle hits the surface at a point that is 15 m horizontally from the point of projection. Find the initial speed of the particle.

4. A particle is projected horizontally at 17 ms\(^{-1}\) from a point, A, which is 48 m above a horizontal surface. Find the horizontal and vertical displacements, from A, of the particle 2 seconds after projection. Hence find the distance of the particle, from A, 2 seconds after projection.

5. A particle is projected horizontally at 10 ms\(^{-1}\). Find the horizontal and vertical components of the particle’s velocity after 1.2 seconds. Hence find the speed and direction of the particle’s velocity after 1.2 seconds.

6. A particle is projected horizontally at 120 ms\(^{-1}\). Find the speed and direction of the velocity of the particle 3 seconds after projection.

7. Two vertical towers stand on horizontal ground. The towers have heights of 50 m and 35 m. A ball is thrown horizontally from the top of the taller tower towards the smaller tower with a speed of 16 ms\(^{-1}\). The ball just clears the smaller tower. Find
   a) the horizontal distance between the towers
   b) the distance from the top of the smaller tower to where the ball first lands.

8. A window in a house is 5.4 m above horizontal ground. A ball is thrown from this window with a speed of 12 ms\(^{-1}\). The ball just clears a vertical wall, whose base is on the horizontal ground. The wall is 8 m from the house. Find the height of the wall and the horizontal distance from the base of the wall to the point where the ball first hits the ground.

9. A batsman strikes a cricket ball horizontally when it is 1 m above the ground. The ball is caught 15 cm off the ground by a fielder standing 10 m from the batsman. Find the speed of the ball as it leaves the batsman’s bat.

10. A bowler releases a cricket ball from a height of 2.1 m above ground. The ball initially travels horizontally and hits the ground subsequently at a horizontal distance of 15 m from the point of release. Find the initial speed of the ball.
1.2 Projection at an angle to the horizontal

Consider a particle that is projected at an angle \( \theta \) above the horizontal with a speed \( u \).

\[ u \]

\[ \theta \]

As in Section 1.1, it is convenient to consider the horizontal and vertical components of the particle's motion separately. However, we must remember that these two components of motion are connected by the common time at any point in space.

We can use trigonometry to determine the magnitude of the particle's initial velocity in both the horizontal and vertical planes.

\[ u_h = u \cos \theta \]

\[ u_v = u \sin \theta \]

\( u_h \) remains constant throughout the flight of the projectile, but \( u_v \) will experience an acceleration due to the force of gravity acting in the direction of this component of motion.

The trajectory – that is, the path – of a projected particle is symmetrical about its highest point.

The time of flight from take off to landing is twice the time taken by the particle to reach its highest point from take off. At the highest point, the particle is instantaneously travelling horizontally and so has no vertical component of speed. The direction of motion of the particle at any other time is found by considering the horizontal and vertical components of the particle's speed. During the first half of the motion, the particle is travelling at an angle **above** the horizontal and, in the second half of motion, the particle is travelling at an angle **below** the horizontal.
Example 5

A golf ball is struck, from a point on horizontal ground, so that its initial speed is 20 ms\(^{-1}\) at an angle of 40° above the horizontal. Find

a) the maximum height the ball reaches

b) the horizontal distance travelled by the ball before it hits the ground

c) the length of time during which the ball is at least 5 m above the ground.

\[ u = 20 \sin 40° \text{ ms}^{-1} \]

\[ v = 0 \text{ at maximum height} \]

\[ s = s \]

\[ t \text{ not required} \]

\[ a = -10 \text{ ms}^{-2} \]

b) Consider vertical motion, where the upward direction is considered positive.

\[ u = 20 \sin 40° \text{ ms}^{-1} \]

\[ v = 0 \text{ at maximum height} \]

\[ s = s \]

\[ t \text{ not required} \]

\[ a = -10 \text{ ms}^{-2} \]

Consider horizontal motion where the speed is constant and equal to 20 cos 40°.

Using \( s = ut \) for horizontal motion gives

\[ x = (20 \cos 40°) \times 2.571150439 \]

\[ x = 39.39231012 \text{ m} \]

\[ x \approx 39.4 \text{ m (3 s.f.)} \]

c) Consider vertical motion, where the upward direction is considered positive.

\[ u = 20 \sin 40° \text{ ms}^{-1} \]

\[ v = 0 \text{ at maximum height} \]

\[ s = s \]

\[ t \text{ not required} \]

\[ a = -10 \text{ ms}^{-2} \]

\[ t = 0.4776744553 \text{ or } t = 2.093475983 \]

Hence the time that the ball is above 5 m is the difference between the two possible values of \( t \).

This is 1.61580152, approximately 1.62 s (3 s.f.).

Obviously we can discard negative time here.
Example 6

A football is kicked, from a point on horizontal ground, so that its initial speed is 10 ms\(^{-1}\) at an angle of 35° above the horizontal towards a wall that is 6 m away. Find

a) the height of the ball above the ground as it hits the wall
b) the speed of the ball at the instant it hits the wall
c) the direction in which the ball is moving at the instant it hits the wall.

a) Consider the horizontal motion of the ball. The ball has to cover a distance of 6 m at a constant speed of 10\cos 35° before it hits the wall.

Using \(s = ut\) for horizontal motion gives

\[
t = \frac{6}{10\cos 35°} = 0.7324647533\text{ s}
\]

when it hits the wall.

Consider vertical motion with the value of \(t\) as above in order to determine the height of the ball above the ground when it hits the wall.

\[
u = 10\sin 35° \text{ ms}^{-1}
\]

\[
v = \text{not required}
\]

\[
s = 0.7324647533\text{ s}
\]

\[
a = -10 \text{ ms}^{-2}
\]

Using \(s = ut + \frac{1}{2}at^2 \Rightarrow\)

\[
s = 10\sin 35° \times (0.7324 \ldots) + \frac{1}{2} \times -10 \times (0.7324 \ldots)^2
\]

\[
s = 1.518722155
\]

\[
s \approx 1.52 \text{ m (3 s.f.)}
\]

b) Find the horizontal and vertical components of the ball’s velocity when it hits the wall, and then use Pythagoras’s Theorem to find the speed.

<table>
<thead>
<tr>
<th>Horizontal motion</th>
<th>Vertical motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u = v = 10 \cos 35° \text{ ms}^{-1})</td>
<td>(u = 10 \sin 35°)</td>
</tr>
<tr>
<td>(s = 6 \text{ m})</td>
<td>(v = v)</td>
</tr>
<tr>
<td>(t = 0.7324647533 \text{ s})</td>
<td>(s \text{ not required})</td>
</tr>
<tr>
<td>(a = 0)</td>
<td>(t = 0.7324647533 \text{ s})</td>
</tr>
<tr>
<td>(a = -10 \text{ ms}^{-2})</td>
<td>(a = -10 \text{ ms}^{-2})</td>
</tr>
</tbody>
</table>
Using $v = u + at$ for vertical motion gives
\[ v = 10 \sin 35^\circ + -10 \times (0.7324647533) \]
\[ = -1.58883169 \text{ ms}^{-1} \]
For horizontal component of motion, $u = v = 10 \cos 35^\circ \text{ ms}^{-1}$.

So, Pythagoras’s Theorem \( V^2 = (10 \cos 35^\circ)^2 + (-1.58883169)^2 \)
\[ V = 8.344193004 \]
\[ V \approx 8.34 \text{ ms}^{-1} \ (3 \text{ s.f.}) \]
Hence the ball has a speed of 8.34 ms\(^{-1}\) at the instant it hits the wall.

c) Using trigonometry, \( \tan \theta = \frac{1.58883169}{10 \cos 35^\circ} \)
\[ \theta = 10.97716959 \]
\[ \theta \approx 11.0^\circ \ (3 \text{ s.f.}) \]
Hence the ball is travelling in a direction which is 11.0\(^\circ\) below the horizontal at the instant it hits the wall.

**Exercise 1.2**

1. A particle is projected at an angle of 30\(^\circ\) above the horizontal from a point on horizontal ground. The initial speed of the particle is 38 ms\(^{-1}\). Find
   a) the time taken for the particle to reach its maximum height
   b) the maximum height of the particle.

2. A ball is thrown with a speed of 9 ms\(^{-1}\) at an angle of 35\(^\circ\) to the horizontal. Find the vertical distance above the point of projection of the particle when it has travelled a horizontal distance of 4 m.

3. A golf ball is struck on level ground so that it travels with a velocity of 30 ms\(^{-1}\) at an angle of 32\(^\circ\) to the horizontal. Find
   a) the time between when the ball was struck and when it first hits the ground
   b) the horizontal distance covered during this time.

4. A stone is thrown from the top of a vertical cliff with a speed of 18 ms\(^{-1}\) at an angle of 25\(^\circ\) above the horizontal. The height of the cliff is 50 m above sea level. The stone lands in the sea. Find
   a) the time that the stone is in the air
   b) the distance from the base of the cliff to the point where the stone lands in the sea
   c) the speed and direction of the stone at the instant it meets the sea.
5. A man kicks a ball from a horizontal surface with a speed of 20 m s\(^{-1}\) so that it makes an angle of 38° with the horizontal. Find how far away from him the ball will land.

6. A football is kicked towards a goal 25 m away with a speed of 20 m s\(^{-1}\) at an angle of 28° to the horizontal. The crossbar is 2.6 m above the pitch. Determine whether the ball passes under or over the bar.

7. A stone is thrown from a point \(P\), 2 m above horizontal ground, with speed 12 m s\(^{-1}\) at an angle of 30° above the horizontal. The stone lands at a point \(Q\) on the ground. Find
   a) the maximum height above the ground the stone reaches
   b) the time the stone takes to travel from \(P\) to \(Q\).

8. A ball is thrown with speed 8 m s\(^{-1}\) at an angle of 50° above the horizontal. Find its speed and direction of motion 0.5 seconds after projection.

9. A ball is thrown with speed 15 m s\(^{-1}\) at an angle 65° above the horizontal. Find the height above the point of projection at which the ball hits a wall 10 m away. Determine whether the ball is rising or falling as it hits the wall.

10. A particle is projected with speed 15 m s\(^{-1}\) at an angle of 40° above the horizontal from a point on horizontal ground. Calculate the time taken for the particle to hit the ground.

11. An athlete is putting the shot from a height of 2 m above horizontal ground. The shot takes 1.6 seconds to land on the ground.
   a) Find the vertical component of the shot’s initial velocity.
   b) If he releases the shot at an angle of 39° above the horizontal, find the horizontal distance that he throws the shot.

12. A particle is projected from a horizontal surface with initial speed 49 m s\(^{-1}\) at an angle of 30° above the horizontal. Find the length of time that the particle is at least 20 m above the surface.

13. A particle is projected from a horizontal plane and has an initial speed of 50 m s\(^{-1}\) at an angle of \(\tan^{-1} \frac{4}{3}\) above the horizontal. Find the length of time that the particle is at least 50 m above the surface.

14. A particle \(P\) is projected with speed 30 m s\(^{-1}\) at an angle of 60° above the horizontal from a point \(O\) on horizontal ground. For the instant when the speed of \(P\) is 17 m s\(^{-1}\) and increasing,
   a) show that the vertical component of the velocity of \(P\) is 8 m s\(^{-1}\) downwards,
   b) calculate the distance of \(P\) from \(O\).