3.1 Vector representation

In Chapter 2, the distinction between scalar and vector quantities was discussed.

Scalar quantities are such things as mass, length, area and speed. Scalar quantities have magnitude (size) only.

Vector quantities are similar to scalars but differ as they have magnitude as well as direction.

In this chapter we look at vectors.

Example 1

The diagram shows a triangle where \( \overrightarrow{OA} = a \), \( \overrightarrow{OB} = b \) and \( M \) is the midpoint of \( AB \).

Express the following vectors in terms of \( a \) and \( b \):

a) \( \overrightarrow{AB} \)

\[ \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -a + b = b - a \]

(the reason for \( -a \) is that we reverse the order: \( AO \) is the opposite of \( OA \))

b) \( \overrightarrow{BA} \)

\[ \overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OB} = -b + a = a - b \]

c) \( \overrightarrow{AM} \)

\[ \overrightarrow{AM} = \frac{1}{2} \overrightarrow{AB} = \frac{1}{2} (b - a) = \frac{b}{2} - \frac{a}{2} \]

d) \( \overrightarrow{BM} \)

\[ \overrightarrow{BM} = \frac{1}{2} \overrightarrow{BA} = \frac{1}{2} (a - b) = \frac{a}{2} - \frac{b}{2} \]

e) \( \overrightarrow{OM} \)

\[ \overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2} \overrightarrow{AB} = a + \frac{1}{2} (b - a) = \frac{a}{2} + \frac{b}{2} \]

An alternative way of writing this solution could have been:

\[ \overrightarrow{OM} = \overrightarrow{OB} + \frac{1}{2} \overrightarrow{BA} = b + \frac{1}{2} (a - b) = \frac{b}{2} + a \]
Example 2

In the parallelogram $OABC$, $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = b$.

$E$ is the point on $CB$ such that $BE : EC = 1 : 3$.

Express the following vectors in terms of $a$ and $b$:

a) $\overrightarrow{CB}$

b) $\overrightarrow{CE}$

c) $\overrightarrow{BE}$

d) $\overrightarrow{OE}$

a) $\overrightarrow{CB} = a$ (as $\overrightarrow{OA}$ and $\overrightarrow{CB}$ are equal)

b) $\overrightarrow{CE} = \frac{3}{4} \overrightarrow{CB} = \frac{3}{4}a$

c) $\overrightarrow{BE} = -\frac{1}{4} \overrightarrow{CB} = -\frac{1}{4}a$

d) $\overrightarrow{OE} = \overrightarrow{OC} + \overrightarrow{CE} = b + \frac{3}{4}a$

Exercise 3.1

1. The diagram shows a triangle $OAB$ with $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. $C$ is the midpoint of $AB$.

Express the following vectors in terms of $a$ and $b$:

a) $\overrightarrow{AB}$

b) $\overrightarrow{BA}$

c) $\overrightarrow{AC}$

d) $\overrightarrow{OC}$

2. The diagram shows a parallelogram $OABC$ with $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = b$.

Express the following vectors in terms of $a$ and $b$:

a) $\overrightarrow{AB}$

b) $\overrightarrow{CB}$

c) $\overrightarrow{OB}$

d) $\overrightarrow{AC}$
3. In parallelogram $OABC$, $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$.
The point $D$ lies on $AB$ such that $AD : DB = 1 : 2$

Express the following vectors in terms of $a$ and $c$:

a) $\overrightarrow{CB}$  
b) $\overrightarrow{BC}$  
c) $\overrightarrow{AB}$  
d) $\overrightarrow{AD}$  
e) $\overrightarrow{OD}$  
f) $\overrightarrow{DC}$

4. $OABC$ is a trapezium with $\overrightarrow{OA} = a$, $\overrightarrow{OC} = c$ and $\overrightarrow{CB} = 3a$. $D$ is the midpoint of $AB$.

Express the following vectors in terms of $a$ and $c$:

a) $\overrightarrow{OB}$  
b) $\overrightarrow{AB}$  
c) $\overrightarrow{OD}$  
d) $\overrightarrow{CD}$

5. The diagram shows a parallelogram $OABC$ with $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$.

$D$ is the point on $AB$ such that $AD : DB = 2 : 3$.

$E$ is the point on $CB$ such that $CE : EB = 3 : 4$.

Express the following vectors in terms of $a$ and $c$:

a) $\overrightarrow{OD}$  
b) $\overrightarrow{EB}$  
c) $\overrightarrow{CD}$  
d) $\overrightarrow{DC}$  
e) $\overrightarrow{ED}$

### 3.2 Resultants

When two or more vectors are added together the single equivalent vector is called the **resultant vector**. We can apply the same process to forces (since they have both magnitude and direction, and can be expressed as vectors.)
Example 3

An anchor is being pulled by two sailors as shown in the diagram. The angle between the two forces is 30°.

Find the magnitude of the resultant of the two forces.
Find also the angle that the resultant makes with the larger force.

To answer this question we look towards the parallelogram of forces.

Two forces $E$ and $F$ are represented by the line segment $AB$ and $AD$, respectively as shown in the diagram.

\[ \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC} \]

In other words, the resultant of the two forces $E$ and $F$, which represent the line segments $\overrightarrow{AB}$ and $\overrightarrow{AD}$, respectively, can be explained by the diagonal $\overrightarrow{AC}$.

$AC$ is the diagonal of our parallelogram $ABCD$ and is called the **parallelogram of forces**.

Going back to our question,

As we know, opposite angles in a parallelogram are congruent.

Looking back at our diagram:

Recall the cosine rule:

\[ c^2 = a^2 + b^2 - 2ab \cos C \]
To find the resultant $\overrightarrow{AC}$ (denoted in the diagram as $R$), we use the cosine rule.

$$R^2 = 60^2 + 40^2 - 2 \times 60 \times 40 \times \cos 150^\circ$$

$$R^2 = 9356.922 \ldots$$

$$R = 96.73 \text{ N (2 d.p.)}$$

The resultant of the two forces is of magnitude 96.73 N.

**Example 4**

A boat is being towed along a canal by cables attached to two horses. The stronger horse produces a force of 300 N, and the other produces 260 N as shown in the diagram.

Find the acute angle between the two forces if their resultant has a magnitude of 540 N.

Using the parallelogram of forces:

By the cosine rule,

$$540^2 = 260^2 + 300^2 - 2 \times 260 \times 300 \times \cos a$$

$$\cos a = \frac{540^2 - (260^2 + 300^2)}{-2 \times 260 \times 300}$$

$$\cos a = -\frac{67}{78}$$

$$a = (180 - 30.798 \ldots) = 149.2016$$

To find the acute angle,

$$360 - 2(149.2016 \ldots) = 61.59676 \ldots$$

We now divide this number by 2 since two of the angles in the parallelogram are the same.

Hence, the angle between the two forces of 300 N and 260 N is 30.8° (1 d.p.)
Exercise 3.2

1. A car has broken down and it is being pulled to the garage by a team of workers. As it is hard work, a pair of workers will pull for 5 minutes each then swap over. On average the forces being applied are 10 N and 12 N, respectively. Find the magnitude and direction of the resultant forces between 10 N and 12 N if the angle between the forces is
   a) 20°  b) 45°  c) 105°

2. Two horses are pulling a cart. Each of the following diagrams shows the forces being applied by the horses at different times. Work out the magnitude of their resultant and the angle it makes with the longer of the two forces.
   a) 210 N
      300 N
   b) 150 N
      200 N
   c) 500 N
      420 N
   d) 160 N
      190 N

3. A tractor has broken down and is being pushed by two people. The magnitude and direction of the force 15° above the horizontal is 210 N. The magnitude and direction of the force 35° below the horizontal is 190 N, as shown in the diagram. Find the resultant of the two forces.

4. Two tug boats are towing a ship into harbour. One of the boats produces a pulling force of 25 000 N. The other produces 27 000 N worth of pulling force. Their resultant has a magnitude of 35 000 N. Find the angle between the two forces and the angle the resultant makes with the larger of the two forces.
5. A young boy is being pulled along on a sledge by both of his parents. In order for the sledge to move along the snow, two forces of 120 N and XN, with an angle of 65° between them, has to be maintained. If the resultant of the two forces has a magnitude of 160 N, find the value of X.

6. A caravan has just been sold at a garage, but the equipment needed to remove it has broken down. Instead, the new owner and the salesperson have decided to push the caravan to the forecourt of the garage so that the owner can connect it to his car and drive it away. If the resultant of the two forces is 620 N, find the angle x between the 450 N force and the horizontal.

![Diagram of a caravan with forces]

### 3.3 Components

Previously we have looked at combining two forces into a single force (called the resultant). There is a reverse process which involves taking a single force and breaking it up into components. These components are also referred to as the resolved parts of the force. The process of finding the resolved parts of a force is called **resolving**.

**Example 5**

Using the diagram, find the components of the given force in the direction of

a) the x-axis
b) the y-axis.

c) Express the force in the form $ai + bj$.

![Diagram of a force with components]

a) Along the x-axis, component $OX = 6 \times \cos 20° = 5.64$ N
b) Along the y-axis, component $OY = 6 \times \sin 20° = 2.05$ N
c) Adding the components along the x and y axes, we get: $(5.64i + 2.05j)$ N
Example 6
Using the diagram, find the components of the given force in the direction of
a) the x-axis
b) the y-axis.
c) Express the force in the form $ai + bj$.

\[ 13 \text{ N} \]

- **a)** Along the x-axis, component $OX = 13 \times \cos 0^\circ = 13 \text{ N}$
- **b)** Along the y-axis, component $OY = 13 \times \sin 0^\circ = 0$
- **c)** Adding the components along the x and y axes, we get: $(13i + 0j) \text{ N} = 13i \text{ N}$

Exercise 3.3
For each of the following diagrams, find the components of the given force in the direction of
a) the x-axis
b) the y-axis.
c) Express the force in the form $ai + bj$.

1. ![Diagram 1](image1)
2. ![Diagram 2](image2)
3. ![Diagram 3](image3)
4. ![Diagram 4](image4)
5. ![Diagram 5](image5)
3.4 Resolving when on an inclined plane

In Section 3.3, we broke a single force up into its components. We are now going to take this idea and to apply it to the situation whereby a mass (given in kg) is on an inclined plane.

Example 7
A body of mass 10 kg rests on an incline of 30°. Find the components of the weight of the body in each of the directions
a) down the plane
b) at right angles to the plane.

In this example, the value of \( g \) is the acceleration of the body due to gravity, and we take its value to be 10 m/s\(^2\). In order to answer the question, we first have to set it up as follows:

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