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1.1 From limits of sequences to limits of functions

Infinity is a concept that has challenged mathematicians and scientists for centuries. Throughout this time the concept of infinity was sometimes denied and sometimes accepted by mathematicians, to the point that it became one of main issues in the history of Mathematics. In the last 150 years, great advances were made: first with the axiomatization of set theory; and then with the work of philosopher Bertrand Russell and his collection of paradoxes. At the climax of all discussions was the work of Georg Cantor on the classification of infinities.

But do infinities really exist? After all, how many types of infinity are there? Does it make sense to compare them and operate with them?

In this chapter we will explore the concept of infinity, starting with an intuitive approach and looking at familiar number patterns: sequences. We will then formalise the idea of ‘the pattern that goes on forever’ and formally define the limit of a sequence. This may help you to better understand the theorems about sequences, although formal treatment of limits of sequences will not be examined. For this reason, all proofs of results have been omitted.

At the end of the chapter we will explore the connections between limits of sequences and limits of functions introduced in the core course. We will also establish criteria for the existence of the limit of a function at a point.
Consider the following numerical sequences:

\[ a_n = 1, 2, 4, 8, 16, \ldots, 2^n, \ldots \]
\[ b_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \]
\[ c_n = -1, 1, -1, 1, -1, 1, \ldots, (-1)^n, \ldots \]

What is happening to the terms of these sequences as \( n \) increases?
Do they approach any real number as \( n \to +\infty \)?

If we graph the sequences \( \{a_n\}, \{b_n\}, \) and \( \{c_n\}, \) we can observe their behaviour as \( n \) increases, and notice that:

- \( \lim_{n \to \infty} a_n = +\infty \) which means that \( \{a_n\} \) diverges;
- \( \lim_{n \to \infty} b_n = 0 \) which means that \( \{b_n\} \) converges to 0;
- \( \lim_{n \to \infty} c_n \) does not exist (it oscillates from 1 to \(-1\)) which means that \( \{c_n\} \) diverges.

The following investigation will help you to better understand what ‘convergent’ means.

**Investigation 1**

1. Use technology to graph the sequence defined by \( u_n = \frac{n+1}{2n+1} \).
2. Hence explain why \( \lim_{n \to \infty} u_n = \frac{1}{2} \).
3. Find the minimum value of \( m \) such that \( n \geq m \Rightarrow |u_n - \frac{1}{2}| < 0.1 \) (i.e. find the smallest integer \( n \) for which the difference between the value \( u_n \) and \( \frac{1}{2} \) is less than 0.1).
4. Consider the positive small quantities \( \varepsilon = 0.01, 0.001, \) and \( 0.0001 \). In each case find the minimum value of \( m \) such that \( n \geq m \Rightarrow |u_n - 0.4| < \varepsilon \).
5. Decide whether or not it is possible to find the order \( m \) such that \( n \geq m \Rightarrow |u_n - 0.4| < 0.1 \). Give reasons for your answer.

Consider now the sequence defined by \( v_n = \left(\frac{-1}{3}\right)^n \).
6. Explain why \( \lim_{n \to \infty} v_n = 0 \).
7. Consider the positive small quantities \( \varepsilon = 0.01, 0.001, \) and \( 0.0001 \). In each case find the minimum value of \( m \) such that \( n \geq m \Rightarrow |v_n| < \varepsilon \).
8. Explain the meaning of \( \lim_{n \to \infty} u_n = L \) in terms of the value of \( |u_n - L| \).
9. Explore further cases of your choice.

You may want to use sequences defined by expressions involving arithmetic and geometric sequences studied as part of the core course.
Definition: Convergent sequences

\{u_n\} is a convergent sequence with \(\lim_{n \to \infty} u_n = L\) if and only if for any \(\varepsilon > 0\) there exists a least order \(m \in \mathbb{Z}^+\) such that, for all \(n \geq m\) \(\Rightarrow |u_n - L| < \varepsilon\).

This definition gives an algebraic criterion to test whether or not a given number \(L\) is the limit of a sequence. However, to apply this test, you must first decide about the value of \(L\).

Example 1

Show that the sequence defined by \(u_n = \frac{3n-1}{n+1}\) is convergent.

Graph the sequence and observe its behavior as \(n\) increases.

Find a simplified expression for \(|u_n - 3|\)

The value of \(m\) is the least positive integer greater than \(\frac{4}{\varepsilon} - 1\)

Use the definition to show that \(\lim_{n \to \infty} u_n = 3\).

Note that this definitions tells you that from the order \(m\) onwards, all the terms of the sequence lie within the interval \([L - \varepsilon, L + \varepsilon]\).

This means that the sequence can have exactly one limit, \(L\).

Useful theorems about subsequences of convergent and divergent sequences:

- If \(\{b_n\} \subseteq \{a_n\}\) is a subsequence of a convergent sequence \(\{a_n\}\), then \(\{b_n\}\) is also a convergent sequence and \(\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n\)

- If \(\{b_n\} \subseteq \{a_n\}\) and \(\{c_n\} \subseteq \{a_n\}\) are subsequences of a sequence \(\{a_n\}\) and \(\lim_{n \to \infty} b_n \neq \lim_{n \to \infty} c_n\) then \(\{a_n\}\) is not convergent (i.e. \(\{a_n\}\) is a divergent sequence).
The following examples show you how to use subsequences of a given sequence to show that the sequence diverges.

**Example 2**

Show that the sequence defined by \( a_n = (-1)^n \cdot 2 \) does not converge.

\[
\begin{align*}
a_n &: -2, 2, -2, 2, \ldots \\
b_n &= a_{2n} = 2 \rightarrow 2 \\
c_n &= a_{2n-1} = -2 \rightarrow -2 \\
\therefore \lim_{n \to \infty} b_n \neq \lim_{n \to \infty} c_n \text{ then } \{a_n\} \text{ is not convergent.}
\end{align*}
\]

Calculate a few terms of the sequence and observe the pattern.

\( \{b_n\} \) and \( \{c_n\} \) are the subsequences of even order terms, and odd order terms, respectively.

If the sequence \( \{a_n\} \) was convergent all its subsequences would have the same limit.

**Example 3**

Show that the sequence defined by \( u_n : \begin{cases} 
  u_1 = 3 \\
  u_{n+1} = -u_n 
\end{cases} \), \( n \in \mathbb{Z}^+ \) is divergent.

Let \( \{v_n\} \) be the subsequence of \( \{u_n\} \) of the terms of even order, i.e. \( v_n = u_{2n} = -3 \) (\( m \) is any positive integer), and let \( \{w_n\} \) be the subsequence of \( \{u_n\} \) of the terms of odd order, i.e. \( w_n = u_{2n-1} = 3 \).

Since \( \lim_{n \to \infty} v_n = -3 \neq 3 = \lim_{n \to \infty} w_n \) the sequence \( \{u_n\} \) cannot converge.

\( \{u_n\} \) and \( \{v_n\} \) are the subsequences of even order terms, and odd order terms, respectively.

3, -3, 3, -3, ...

If the sequence \( \{u_n\} \) was convergent all its subsequences would have the same limit.

We can also use subsequences to determine the limit of a convergent sequence defined recursively as we will see later in this chapter.

**Exercise 1A**

1. Consider the sequence defined by \( u_n = \frac{n+3}{2n+1} \). Find the least value of \( m \in \mathbb{Z}^+ \) such that \( n \geq m \implies |u_n - \frac{1}{2}| < 0.001 \).

2. Consider the sequence defined by \( v_n = \frac{n+1}{3n-1} \).
   
   a. Graph the sequence and, if possible, state its limit.
   
   b. Find the least value of \( m \in \mathbb{Z}^+ \) such that \( n > m \implies |v_n - \frac{1}{3}| < 0.001 \).

3. Consider the sequence defined by \( u_n = \frac{4^n - 3}{4^n} \).
   
   a. Graph the sequence and, if possible, state its limit.
   
   b. Find the least value of \( m \in \mathbb{Z}^+ \) such that \( n > m \implies |u_n - 1| < 0.0005 \).