4 The finite in the infinite

Exercise 4A

1. Let $\lim_{n \to \infty} 3^n \neq 0$, hence the series diverges.

2. Let $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with $r = \frac{1}{3}$, hence it converges to its sum, $S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1.5$.

3. Let $\lim_{n \to \infty} \frac{2n!}{3n+3} = \lim_{n \to \infty} \frac{2}{3 + \frac{3}{n!}} = \frac{2}{3} \neq 0$, hence the series diverges.

4. Let $\sum_{n=0}^{\infty} 3 \left(\frac{1}{7}\right)^n = 3.7 + \sum_{n=1}^{\infty} 3 \left(\frac{1}{7}\right)^n$ is a geometric series with $r = \frac{1}{7}$, hence it converges to its sum, $S_\infty = 3 \cdot \frac{1}{1-\frac{1}{7}} = \frac{21}{6} = 3\frac{1}{2}$.

Hence $S = 24.5$

5. Let $\sum_{n=1}^{\infty} \frac{e^n}{\pi^n} = \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ is a geometric series with $r = \frac{e}{\pi} \approx 1.03$. Since $r > 1$ the series does not converge.

6. Let $\frac{3}{2} + \frac{6}{3} + \frac{9}{4} + \frac{12}{5} + \ldots = \sum_{n=1}^{\infty} \frac{3n}{n+1 - \frac{3n}{n+1}} = \lim_{n \to \infty} \frac{3}{1 - \frac{3}{n}} = \frac{3}{1} \neq 0$, hence series diverges.

7. Let $\sum_{n=1}^{\infty} \frac{3}{2^n-1} = \sum_{n=1}^{\infty} \frac{6\sum_{n=2}^{\infty} \frac{1}{2^n}}{\sum_{n=2}^{\infty} \frac{1}{2^n}}$ is an infinite geometric series, $r = \frac{1}{2}$, hence $S_\infty = \frac{2}{1-\frac{1}{2}} = 2 \implies S_\infty = 6$.

8. Let $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$. Both series are geometric with $r = \frac{1}{2}$ and $r = \frac{1}{3}$ respectively. Hence $S_n = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 2 - \frac{3}{2} = 0.5$.

Exercise 4B

1. Corollary 1: $\int_a^b f(t) \, dt = -\int_b^a f(t) \, dt$

By the FTC, $g(x) = \int_a^b f(t) \, dt$, $b \leq x \leq a$ is an anti-derivative of $f$, i.e., $g'(x) = f(x)$, $b < x < a$

$$\implies -\int_b^a f(t) \, dt.$$

Corollary 2: $\int_a^b f(x) \, dx = F(a) - F(b)$

Let $g(x) = \int_a^b f(t) \, dt$, $a \leq x \leq b$. If furthermore $F$ is any anti-derivative of $f$, then by the mean value theorem, $F(x) = g(x) + c$, for some constant $c$.

Evaluating therefore $F(b) - F(a)$,

$$F(b) - F(a) = [g(b) + c] - [g(a) + c]$$

$$= g(b) - g(a)$$

$$= \int_a^b f(t) \, dt - \int_b^a f(t) \, dt$$

$$= \int_a^b f(t) \, dt - 0$$

$$= \int_a^b f(t) \, dt$$
\( x = \frac{1}{2} \int_n^{\pi/2} \frac{\sin(t)}{1 + \cos(t)^2} \, dt \)
Exercise 4D

1 From the GDC, \( f(x) = \frac{3}{x + 1} \) is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{3}{x + 1} \, dx = \lim_{b \to \infty} \int_1^b \frac{3}{x + 1} \, dx = 3 \lim_{b \to \infty} \left[ \ln(x + 1) \right]_1^b
\]
\[
= 3 \lim_{b \to \infty} (\ln(b + 1) - 2) = 3 \lim_{b \to \infty} \ln \left( \frac{b + 1}{4} \right) = \infty
\]

hence the series diverges.

2 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_1^b \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \left[ \ln(x^2 + 1) \right]_1^b
\]
\[
= \frac{1}{2} \lim_{b \to \infty} \left[ \ln(b^2 + 1) - \ln 2 \right] = \infty,
\]
hence the series diverges.

3 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \left[ \arctan x \right]_1^b
\]
\[
= \lim_{b \to \infty} \left[ \arctan b - \arctan 1 \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\]
hence the series converges.

4 The integral test cannot be applied since the function is increasing for \( x \geq 1 \). However, since \( \lim_{b \to \infty} \sin \left( \frac{1}{x} \right) = 1 \neq 0 \), the series diverges by the nth test for divergence.

5 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} e^{\frac{x}{x^2 - 1}} \, dx = \lim_{b \to \infty} \int_1^b e^{\frac{x}{x^2 - 1}} \, dx = \lim_{b \to \infty} \left[ \arctan e^{\frac{x}{x^2 - 1}} \right]_1^b
\]
\[
= \lim_{b \to \infty} \left[ \arctan e \right] = \frac{\pi}{2}
\]
Hence, the series converges.

6 The integral test cannot be applied since the function is not decreasing for all \( x \geq 1 \).

7 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{4x}{(2x^2 + 3)^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{4x}{(2x^2 + 3)^2} \, dx = \lim_{b \to \infty} \left[ \frac{1}{2x^2 + 3} \right]_1^b
\]
\[
= \lim_{b \to \infty} \left( \frac{1}{2b^2 + 3} + \frac{1}{5} \right) = \frac{1}{5}
\]
Hence, the series converges.

8 The integral test cannot be applied since the function is not decreasing for all \( x \geq 1 \).

9 The integral test cannot be applied since the function is not decreasing for all \( x \geq 1 \).

10 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \left[ \ln(\ln(x)) \right]_1^b
\]
\[
= \lim_{b \to \infty} (\ln(\ln(b)) - \ln(\ln(1))) = \infty
\]

Hence the series diverges.

11 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \left[ \arctan x \right]_1^b
\]
\[
= \lim_{b \to \infty} \left[ \arctan b - \arctan 1 \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\]
Hence the series converges.

12 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 1 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{\ln x}{x^3 + 1} \, dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x^3 + 1} \, dx = \lim_{b \to \infty} \left[ -\frac{x + 1}{x} \right]_1^b
\]
\[
= \lim_{b \to \infty} \left( -\frac{b + 1}{b} + \frac{ln 2}{2} \right) = \frac{ln 2}{2}
\]
Hence the series converges.

13 From the GDC, the function is continuous, positive and decreasing for all \( x \geq 2 \), hence we can apply the integral test.

\[
\int_1^{\infty} \frac{x^2}{e^x} \, dx = \lim_{b \to \infty} \int_1^b \frac{x^2}{e^x} \, dx = \lim_{b \to \infty} \left[ -e^{-x} (x^2 + 2x + 2 \right]_1^b
\]
\[
= \lim_{b \to \infty} \left( -e^{-b} (b^2 + 2b + 2) + \frac{5}{e} \right) = \frac{5}{e}
\]
Hence the series converges.

Exercise 4E

1 Using the p-series test, \( p = \pi, \pi > 1 \), hence the series converges.

2 Using the p-series test, \( p = \frac{\pi}{4}, \frac{\pi}{4} < 1 \), hence the series diverges.

3 \( 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + ... = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \)

Using the p-series test, \( \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \frac{1}{n^2}, p = \frac{3}{2}, \frac{3}{2} > 1 \), hence the series converges.

4 \( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

Using the p-series test, \( p = \frac{1}{2}, \frac{1}{2} < 1 \) hence the series diverges.

5 \( p = \frac{1}{2}, \frac{1}{2} < 1 \), hence the series diverges.
Exercise 4F

1. \[ \frac{1}{n^2 + 1} < \frac{1}{n^2} \], for \( n \geq 1 \). Since \( \frac{1}{n^2} \) converges by the p-series test, the original series also converges.

2. \[ \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n^2}} = \frac{1}{n} \], for \( n \geq 2 \). Since \( \frac{1}{n} \) diverges by the p-series test, the original series also diverges.

3. \[ \frac{1}{(n + 1)!} < \frac{1}{n^2} \], for \( n \geq 1 \). Since \( \frac{1}{n^2} \) converges by the p-series test, the original series converges.

4. \( \ln(n) \) is convergent for \( n \geq 2 \), hence \( \frac{1}{\ln(n)} \geq \frac{1}{n} \), for \( n \geq 2 \). Since \( \frac{1}{n} \) diverges by the p-series test, the original series diverges.

5. \[ \frac{1}{\sqrt{n^2} + 1} < \frac{1}{\sqrt{n^2}} = \frac{1}{n} \], for \( n \geq 1 \). Since \( \frac{1}{n} \) converges by the p-series test, the original series converges.

6. \( \ln(n) \) is convergent for \( n \geq 3 \), hence \( \frac{1}{\ln(n)} \geq \frac{1}{n} \), for \( n \geq 3 \). Since \( \frac{1}{n} \) diverges by the p-series test, the original series diverges.

7. \( \frac{\cos^2 n}{n^2} \) is convergent for \( n \geq 1 \). Since \( \frac{1}{n^2} \) converges by the p-series test, the original series converges.

8. \[ \frac{n - 1}{n^2 \sqrt{n}} = \frac{n - 1}{n^2} \times \frac{1}{\sqrt{n}} \], for \( n \geq 1 \). Since \( \frac{1}{n^2} \) converges by the p-series test, the original series converges.

9. \( \arctan(x) \leq \frac{x}{2} \), hence \( \frac{1}{\sqrt{n^2 + 1}} < \frac{1}{\sqrt{n}} \) for \( n \geq 1 \). Hence

\[ \sum \frac{1}{\sqrt{n^2 + 1}} < \frac{1}{\sqrt{n}} \times \sum \frac{1}{\sqrt{n}} \]. Since \( \frac{1}{\sqrt{n}} \) converges by the p-series test, the original series also converges.

10. \( \ln(n) \) is convergent for \( n \geq e^2 \), hence \( \frac{1}{\ln(n)} \leq \frac{1}{n} \). Since \( \frac{1}{n} \) converges by the p-series test, the original series converges.

Exercise 4G

1. Compare with \( \sum \frac{1}{\sqrt{n}} \). Hence,

\[ \lim_{n \to \infty} \frac{\sqrt{n}}{2^n} = \lim_{n \to \infty} \frac{n^{1/2}}{2^n} = \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 1 \], and the series diverges since \( \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n} \) diverges.

2. Compare with \( \sum \frac{n^2}{n^3} \). Hence,

\[ \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \left( \frac{n}{n} \right)^2 = \lim_{n \to \infty} \frac{n}{n+1} = \frac{1}{2} \]. The series converges since the p-series test, \( \sum \frac{1}{n^2} = \sum \frac{1}{n^3} \) converges.

3. Compare with \( \sum \frac{1}{n} \). Hence \( \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{n}{n + \sqrt{n}} = 1 \).

The series diverges, since \( \sum \frac{1}{n} \) diverges.

4. Compare with \( \sum \frac{1}{2^n} \), a convergent geometric series.

Since \( \lim_{n \to \infty} \frac{2^n}{2^n - 1} = 1 \), the series converges.

5. Compare with \( \sum \frac{n^2}{\sqrt{n^3}} \).

\[ \lim_{n \to \infty} \frac{n^3}{\sqrt{n^3}} = \lim_{n \to \infty} \frac{3n^2 + 2n}{n^3} = \lim_{n \to \infty} \frac{n^2}{4 + n^3} = 3 \]. Since \( \sum \frac{n^2}{\sqrt{n^3}} = \sum \frac{1}{n^2} \), the series converges.

6. \( \sum \left( \frac{1}{2^n - 1} - \frac{1}{2^n} \right) = \sum \frac{1}{2^n} \). Compare with \( \sum \frac{1}{n^2} \).

\[ \lim_{n \to \infty} \frac{1}{2^n - 1} = \lim_{n \to \infty} \frac{n^2}{4^n - 2n} = \frac{1}{4} \]. Hence, since \( \sum \frac{1}{n^2} \) converges by the p-series test, the series converges.

7. Compare with \( \sum \frac{1}{n^{1/3}} \).

\[ \lim_{n \to \infty} \frac{1}{n^{1/3} \sqrt[n]{n+3}} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{n^{1/3} \sqrt[n]{n+3}} = \frac{1}{3} \]. Hence, since \( \sum \frac{1}{n^{1/3}} \) diverges by the p-series test, the series diverges.

8. Compare with \( \sum \frac{n^2}{n^3} \). Which diverges by the \( n \)th term test for divergence, since \( \lim_{n \to \infty} \frac{2^n}{n^3} \neq 0 \).

\[ \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1 + \frac{n}{n}}{3} = \frac{1}{3} \]. Hence the series diverges.

Exercise 4H

1. \[ \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^2 = 2 \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 2 \).

Since \( L > 1 \) the series diverges.

2. \[ \lim_{n \to \infty} \frac{1}{n^2 + 1} = \lim_{n \to \infty} \frac{n^2}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0 \).

Since \( L < 1 \) the series converges.

3. \[ \lim_{n \to \infty} \frac{1}{n^2 + 1} = \lim_{n \to \infty} \frac{n^2}{3^n \cdot (n+1)!} = 3 \lim_{n \to \infty} \frac{1}{n+1} = 0 \).

Since \( L < 1 \) the series converges.
Exercise 4I

1 Using the ratio test,
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^2 = \frac{1}{2} \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = \frac{1}{2}.
\]
Since \( L < 1 \) the series converges by the ratio test.

2 The series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) has terms of different signs, since \(-1 \leq \sin n \leq 1\). Since \( \left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2} \) for all \( n \), and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the p-series test, the series converges absolutely, therefore the series converges.

3 Using the ratio test,
\[
\lim_{n \to \infty} \frac{n!}{2\pi n} = 2 \lim_{n \to \infty} \left( \frac{n}{n+1} \right)! = 2 \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 0.
\]
Since \( L < 1 \) the series converges by the ratio test.

4 Using the comparison test, \( \lim_{n \to \infty} \frac{\arctan n}{n^3} \leq \frac{\pi}{2} \) for all \( n \geq 1 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges by the p-series test, so does \( \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3} \). Hence the series converges absolutely.

5 Using the ratio test,
\[
\lim_{n \to \infty} \frac{1}{2n} = \lim_{n \to \infty} \frac{2n!}{2(n+1)!} = \frac{1}{2} \lim_{n \to \infty} \frac{1}{n+1} = 0.
\]
Since \( L < 1 \), the series converges. Hence the series converges absolutely.

6 Since \( \lim_{n \to \infty} \frac{n!}{e^n} \neq 0 \), the series does not converge absolutely.

Exercise 4J

1 a The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges.

For the series of absolute values, \( \ln(n) \leq n \) for \( n \geq 2 \), hence \( \frac{1}{\ln n} \geq \frac{1}{n} \). Since \( \frac{1}{n} \) diverges by the p-series test, the series of absolute values diverges by the comparison test. Hence the series converges conditionally.

b The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. Using the ratio test for the series of absolute values,
\[
\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{1}{n+1} = \frac{1}{n+1} \frac{n+1}{n+2} = \frac{1}{n+2}.
\]
Since \( L < 1 \), the series converges absolutely.

c The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. The series of absolute values diverges by the p-series test, hence the series converges conditionally.

d The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. For the series of absolute values, using the ratio test,
\[
\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{1}{n+1} = \frac{1}{n+1} \frac{1}{n} = 0.
\]
Since \( L < 1 \), the series converges absolutely.

e The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. The series of absolute values converges because it is a geometric series, with \( |r| < 1 \). Hence the series converges absolutely.

f The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. The series of absolute values converges because it is the harmonic series, which diverges. Hence the series converges conditionally.

The integral test can be used on the series of absolute values, since \( f(x) = \frac{1}{x \ln x} \) is continuous, positive and decreasing for all \( x \geq 1 \).

Hence, \( \int_{1}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} [\ln(\ln x)]_{1}^{b} = \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 1)] = \infty. \)
Since the series of absolute values diverges, the series converges conditionally.

h The absolute value of the terms is decreasing, and \( \lim u_n = 0 \), hence the series converges. The series of absolute values converges by comparison with \( \sum \frac{3}{n^2} \), hence the series converges absolutely.
The conditions for Leibniz’ theorem are met since the absolute value of the terms is decreasing, and \( \lim_{n \to \infty} u_n = 0 \). Hence the truncation error is
\[
\| u_{n+1} \| \leq 0.001 \Rightarrow \frac{1}{(n+1)!} \leq 0.001 \Rightarrow n = 5.
\]
Hence, \( S_5 \approx 0.948 \).

The conditions for Leibniz’ theorem are met since the absolute value of the terms is decreasing, and \( \lim_{n \to \infty} u_n = 0 \), hence the series converges.

Since the series begins with \( n = 2 \), and we want the first three terms, i.e., \( n = 2, 3, 4 \), therefore
\[
u_2 = \frac{1}{5}, \quad u_3 = \frac{1}{35}, \quad u_4 = -0.012644.
\]

The conditions for Leibniz’ theorem are met since the absolute value of the terms is decreasing, and \( \lim_{n \to \infty} u_n = 0 \).

Exercise 4K

1 Using the ratio test,
\[
\lim_{n \to \infty} \frac{6^{n+1}}{6^n} = \lim_{n \to \infty} \left( \frac{6^{n+1}}{6^n} \right) = 6 \lim_{n \to \infty} \frac{1}{n+1} = 6 \cdot 0 = 0.
\]
Since \( L < 1 \), the series converges.

2 Using the ratio test,
\[
\lim_{n \to \infty} \frac{3^{n+1}}{n^2} = \lim_{n \to \infty} \left( \frac{3^{n+1}}{n^2} \right) = \frac{2}{3} \lim_{n \to \infty} \frac{n+1}{n} = \frac{2}{3}.
\]
Since \( L < 1 \), the series converges.

3 This is a geometric series whose common ratio is \( \frac{1}{e} \). Since \( |r| < 1 \), the series converges to its sum.

4 Using the GDC, we see that the function that represents the series is continuous, positive and decreasing for all \( n \geq 1 \). Hence, we can use the integral test. Using the substitution \( u = \frac{1}{x} \),
\[
\int_1^\infty \frac{1}{x^3} \tan \left( \frac{1}{x} \right) dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^3} \tan \left( \frac{1}{x} \right) dx
= \lim_{n \to \infty} \left( \frac{1}{\cos x} \right)_{b} = -\ln(\cos(1)).
\]
Hence the series converges.

5 Using the limit comparison test with \( \sum_{n=1}^{\infty} \frac{2n+1}{n} \),
\[
\lim_{n \to \infty} \frac{2n^2 + 3n - 1}{n} = \lim_{n \to \infty} \frac{2n^2 + n}{2n^2 + 3n - 1} = 1.
\]
Since \( L \) is a positive real number, the series diverges since the harmonic series diverges.

6 Since the absolute value of the terms of the series is decreasing, and \( \lim_{n \to \infty} u_n = 0 \), the series converges.
For all \( n \geq 1 \), \( \sqrt{3n} > \sqrt{3n-1} \), hence \( \frac{1}{\sqrt{3n}} < \frac{1}{\sqrt{3n-1}} \).
Since \( \sum \frac{1}{\sqrt{n}} \) diverges by the p-series test, so does \( \sum \frac{1}{\sqrt{n}} \). Hence the series does not converge absolutely, and converges conditionally.

7 \( \sum \frac{2}{\ln n} = \sum \frac{2}{2\ln n} = \sum \frac{1}{\ln n} \).
Since \( \frac{1}{n} < \frac{1}{\ln n} \) for all \( n \geq 2 \), the series diverges by comparison with the harmonic series.

8 Since \( \lim_{n \to \infty} \frac{n(n+1)}{n^2 + 5n} \neq 0 \), the series diverges by the Divergence Test.

9 Since the absolute value of the terms of the series is decreasing, and \( \lim_{n \to \infty} u_n = 0 \), the series converges.
Using the integral test, since the function representing the series is positive, decreasing and continuous for all \( x \geq 2 \),
\[
\int_2^\infty \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} dx
= \lim_{b \to \infty} \left[ 2 \ln x \right]_2^b
= \lim_{b \to \infty} (2\ln b - 2\ln 2) = \infty.
\]
Hence, the series diverges absolutely, and converges conditionally.

10 Using the ratio test,
\[
\lim_{n \to \infty} \frac{4^{n+1} (n+1)^2}{4^n n^2} = \lim_{n \to \infty} \left( \frac{4^{n+1} (n+1)^2}{4^n n^2} \right) \times \frac{n!}{4^n n^2}
= \lim_{n \to \infty} \frac{4(n+1)^2}{n^2} = 0.
\]
Since \( L < 1 \), the series converges.
11 Using the ratio test,
\[ \lim_{n \to \infty} \frac{(n+1)!}{n!} \frac{2 \cdot 5 \cdot 8 \ldots (3n+2)}{2 \cdot 5 \cdot 8 \ldots (3n+2)} = \lim_{n \to \infty} \frac{(n+1)!}{3n+5} = \frac{1}{3} \]
Since \( L < 1 \) the series converges.

12 \[ \sum_{n=1}^{\infty} \frac{(-2)^n}{n^a} \] \( \sum_{n=1}^{\infty} \frac{4^n}{n^a} \). Using the ratio test,
\[ \lim_{n \to \infty} \frac{4^{n+1}}{4^n} \cdot \frac{n}{(n+1)(n+1)^{a-1}} = \lim_{n \to \infty} \frac{4n^n}{4^n (n+1)^{a-1}} = 0. \]
Hence, the series converges. (This one is best done with the Root Test, which is not on the syllabus.)

13 Using the limit comparison test with \( \sum \frac{1}{n\sqrt{n}} \),
\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n\sqrt{n}}} = \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1, \text{ and } 1 > 0, \text{ hence the series converges since } \sum \frac{1}{n\sqrt{n}} \text{ converges by the p-series test.} \]

14 \[ 0 < \frac{\arctan n}{n^2} < \frac{\pi}{4} \sum \frac{\pi}{2} = \frac{\pi}{2} \sum \frac{1}{n^2} \] Since this series converges by the p-series test, the original series also converges.

15 a The series is continuous and decreasing, but not positive for all \( n \geq 1 \).
b The series does not have all non-negative terms.
c The series does not have all non-negative terms.
d i the sequence of absolute values is not decreasing.
ii the limit of the nth term as \( n \) toes to infinity does not equal 0.

16 a \[ \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \]
\[ S_1 = \frac{1}{2!}, S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}, S_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{23}{24}, \]
\[ S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{119}{120}, S_5 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} = \frac{719}{720}, \]
\[ S_6 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \frac{6}{7!} = \frac{5039}{5040} \]
Conjecture: \( S_n = \frac{(n+1)! - 1}{(n+1)!} \)

b Proof by Mathematical Induction:
\[ P_n : S_n = \frac{(n+1)! - 1}{(n+1)!}, n \in \mathbb{Z}^+. \]
For \( n = 1 \), show that \( \frac{(n+1)! - 1}{(n+1)!} = \frac{n}{(n+1)!} \).
LHS: \( \frac{(1+1)! - 1}{(1+1)!} = \frac{2! - 1}{2!} = \frac{1}{2!} = \text{ RHS} \)
Assume \( P_k \) is true, i.e., \( S_k = \frac{(k+1)! - 1}{(k+1)!}, k \in \mathbb{Z}^+ \).
We want to show that \( P_{k+1} \) is true, i.e.,
\[ S_{k+1} = S_k + T_{k+1} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} \]
by assumption
\[ = \frac{(k+2)! - (k+1)! - 1 + k + 1}{(k+2)!} = \frac{(k+2)!}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!} = P_{k+1} \]
Since \( P_n \) is true for \( n = 1 \), and proven true for \( n = k+1 \) when \( n = k \), \( k \in \mathbb{Z}^+ \), then by mathematical induction \( P_{k+1} \) is true.

b Using the ratio test,
\[ \lim_{n \to \infty} \left[ \frac{(n+1)!}{(n+2)!} \right] = 0. \]
Since \( L < 1 \), the series converges.

*Finding the sum is best done after Chapter 5 using \( f(x) = e^x \) and Taylor series.
Let \( S \) denote \( \sum_{n=1}^{\infty} \frac{1}{n!} \). Then
\[ S + e - 2 = S + \sum_{n=1}^{\infty} \frac{1}{n!} - 1 = S + \frac{1}{(n+1)!} \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1. \]
Hence, \( S = 1 \).

Review Exercise

1 a Using the nth term test for divergence,
\[ \lim_{n \to \infty} \left( \frac{n}{n+4} \right)^n = \lim_{n \to \infty} \left( \frac{1}{1 + \frac{4}{n}} \right)^n = \frac{1}{e} \neq 0, \text{ hence the series diverges.} \]

b Using the limit comparison test with \( \sum \frac{1}{n} \),
\[ \lim_{n \to \infty} \frac{1}{1 + \frac{4}{n}} \]
harmonic series, the series diverges.

c For \( n \geq 2 \), and since \( 2 < e, \ln(n) < n - 1 \). Hence,
\[ \frac{1}{\ln n} > \frac{1}{n - 1} \Rightarrow \sum \frac{1}{\ln n} > \sum \frac{1}{n-1} = \sum \frac{1}{n}. \]
Since the harmonic series diverges, by the comparison test the given series will also diverge.
By the ratio test, 
\[ \lim_{n \to \infty} \left( \frac{n+1}{n} \right) \left( \frac{e^n}{e^n} \right) = 1 < 1. \] 
Hence the series converges.

Since \( \cos n\pi = (-1)^n \),
\[ \frac{(n+10)\cos n\pi}{n^{1/4}} = (-1)^n \frac{(n+10)}{n^{1/4}}. \]
Since \( \lim \frac{(n+10)}{n^{1/4}} = 0 \) and the sequence is decreasing, the given series is convergent by the alternating series test.

Hence the series converges.

By the ratio test, 
\[ \lim_{n \to \infty} \left( \frac{k!(k+1)!}{k!} \right) = \lim_{n \to \infty} k = 0. \]
Hence the series converges. 
\[ \sum_{k=1}^{\infty} \frac{1}{k} \]
Furthermore, 
\[ S = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \ldots \]
\[ = 1 - \left( \frac{1}{1!} \right) - \left( \frac{1}{2!} \right) - \left( \frac{1}{3!} \right) - \ldots \]
\[ = 1. \]

\[ \lim_{n \to \infty} \frac{1}{(n+1)} = 0, \]
and for all \( n, a_n \geq a_{n+1}, \) hence the series converges by the alternating series test 4.

The sum of the series therefore, correct to 6d.p., is 
\[ 1 - \frac{1}{2^2} - \frac{1}{3^2} - \ldots + \frac{1}{7^2} = 0.992594 \]

By comparison with the harmonic series, 
\[ \lim_{n \to \infty} \left( \frac{n+n^2}{1+n^n} \right) = \lim_{n \to \infty} \frac{n}{n^n} = 1. \]
Hence the series diverges by the limit comparison test.

Since \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \Rightarrow a_n < 1 \) for large \( n \).

By comparison, and since \( \sum a_n b_n \) converges, then \( \sum a_n b_n \) converges. Similarly, \( b_n < 1 \Rightarrow b_n^2 < b_n \), and hence \( \sum b_n^2 \) converges because \( \sum b_n \) converges.

\[ u_n = \frac{3n+2}{2^{n+1}}. \]
Using the ratio test, 
\[ \lim_{n \to \infty} \left( \frac{3n+2}{3n-2} \right) \left( \frac{2^{n+1}}{2^n} \right) = 2 \frac{1}{3} \lim_{n \to \infty} \frac{3n+2}{3n-2} = 2 \frac{1}{3} < 1. \]
Hence the series converges.

\[ u_n = \frac{2n-1}{(\sqrt{2})^n}. \]
Then by the ratio test, 
\[ \lim_{n \to \infty} \left( \frac{2n+1}{2n-1} \right) \left( \frac{\sqrt{2}}{\sqrt{2}} \right)^n = \frac{1}{\sqrt{2}} \lim_{n \to \infty} 2n+1 = \frac{1}{\sqrt{2}} < 1. \]
Hence the series converges.

Using the limit comparison test with \( b_n = \frac{1}{n^2} \),
\[ \lim_{n \to \infty} \frac{1 - \cos \left( \frac{1}{n} \right)}{\frac{1}{n^2}} = 0 \]
which allows us to apply L’Hopital’s rule. Taking the derivative of numerator and denominator, we obtain
\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n^2} + \frac{1}{n^3}} = \frac{1}{2} \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{1} = \frac{1}{2} \lim_{n \to \infty} \frac{n}{1} = \frac{1}{2}. \]
Since \( \sum \frac{1}{n^2} \) converges by the p-series test, the given series also converges.

The sum of the areas of the smaller rectangles is \( \sum_{n=1}^{\infty} \frac{1}{n^p} \).

The sum of the areas of the larger rectangles is \( \sum_{n=1}^{\infty} \frac{1}{n^p} \).

Since the area under the curve is between smaller areas and the larger areas, 
\[ \sum_{n=1}^{\infty} \frac{1}{n^p} < \int_{1}^{\infty} \frac{1}{x^p} \, dx < \sum_{n=1}^{\infty} \frac{1}{n^p} \]

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx = \left[ \frac{x^{-p+1}}{-p+1} \right]_{1}^{\infty} = \frac{1}{p-1}. \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} - 1 < 1 + \int_{1}^{\infty} \frac{1}{x^p} \, dx = 1 + \frac{1}{p-1} = \frac{p}{p-1}. \]

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx < \frac{p}{p-1} \text{ and } \int_{1}^{\infty} \frac{1}{x^p} \, dx < \sum_{n=1}^{\infty} \frac{1}{n^p}. \]

Hence, 
\[ \frac{1}{p-1} < \sum_{n=1}^{\infty} \frac{1}{n^p} < \frac{p}{p-1}. \]
The total area of the upper rectangles is
\[ \frac{1}{n^4} \times 1 + \frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \ldots = \sum_{i=n}^{\infty} \frac{1}{i^4} . \]
The total area of the lower rectangles is
\[ \frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \frac{1}{(n+3)^4} \times 1 + \ldots = \sum_{i=n}^{\infty} \frac{1}{i^4} . \]
The total area under the curve from \( x = n \) to infinity lies between the above two sums, hence
\[ f \left[ x^n \right]_{x=n}^{\infty} \approx N = \frac{\pi^4}{90} \approx 90.0268 \implies N = 90. \]

(i) When \( n = 8 \), \( S < 1.08243 \ldots \) and \( S > 1.08219 \ldots \), hence \( S = 1.082 \) to 3 d.p.

(ii) Substituting this value for \( S \),
\[ N \approx \frac{\pi^4}{90} \approx 90.0268 \implies S = 1.082 \]

10 (a) \( S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{2n} + \ldots + \frac{1}{2n} > S_n + \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = S_n + \frac{1}{2} . \)

Replacing \( n \) by \( 2n \), \( S_{4n} > S_{2n} + \frac{1}{4} > S_n + 1 . \)

Continuing this way, \( S_{8n} > S_n + \frac{3}{4} \). And in general, \( S_{2^{m}n} > S_n + \frac{m}{2} . \) Then, letting \( n = 2 \),
\[ S_{2^{m+1}} > S_2 + \frac{m}{2} . \]

(c) \( S_{2^{m+1}} > N \) if \( S_2 + \frac{m}{2} > N \), or if \( m > 2(N - S_2) \). Hence the sequence is divergent.

11 (i) The sequence of absolute values is decreasing and the \( n \)th term tends to 0 as \( n \) tends to infinity, hence by the alternating series test, the series converges.

(ii) The partial sums are 0.333, 0.111, 0.269, 0.148, 0.246. \( S_n \) lies between any pair of successive partial sums, hence it lies between 0.148 and 0.246, therefore it is less than 0.25.

12 (a) The sequence of absolute values is decreasing and the \( n \)th term tends to 0 as \( n \) tends to infinity, hence by the alternating series test, the series converges.

(b) \( S_4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = 0.841468(6 \text{d.p.}) \)

\( S_4 \) error \( < a_5 \), hence
\[ \text{error} < \frac{1}{9!} \text{ or error} < 0.00000276 . \]