Ancient mathematics and modern methods

Answers

Skills check

1 Using Δ’s CFE and CBA, \( \frac{6.5}{6.5} \div \frac{CF}{CB} \)

Using Δ’s CDF and BAF, \( \frac{4}{4} \div \frac{CF}{CB} \)

\( \frac{CB - CF}{CF} \div \frac{6.5}{4} \Rightarrow \frac{CB - CF}{CF} - 1 = 1.625 \Rightarrow \frac{CB - CF}{CF} = 2.265 \)

∴ \( \frac{EF}{6.5} = \frac{1}{2.265} \Rightarrow \frac{EF}{51}{\div} \frac{21}{} = 2.48 \text{ m} \)

Exercise 8A

1 a \( \hat{BAC} = 90° - 28° = 62° \)

\( \sin 28° = \frac{AB}{8} \therefore AB = 8\sin 28° = 3.76 \text{ cm} \)

\( \cos 28° = \frac{BC}{8} \therefore BC = 8\cos 28° = 7.06 \text{ cm} \)

b \( QR^2 = 7^2 - 4.2^2 \therefore QR = \sqrt{31.36} = 5.6 \text{ cm} \)

\( \sin \hat{PQR} = \frac{4.2}{7} \therefore \hat{PQR} = 36.9° \)

\( Q\hat{PR} = 90° - 36.9° = 53.1° \)

2 \( \tan \alpha = \frac{BT}{AB} \therefore BT = 30\tan 52.3° = 38.8 \text{ m} \)

3 \( \cos \theta = \frac{r}{15} \therefore r = 15\cos 1.23 = 5.013… \)

\( \text{arc length} = 2\pi = 31.50… \)

∴ \( \hat{\varphi} = 31.50… \therefore \varphi = 2.10 \text{ radians} \)

4 a \( \cos \alpha = \frac{0.3}{1.5} \therefore \alpha = 1.369\ldots \)

\( 2\alpha = 2.738\ldots \)

\( \theta = 2\pi - 2.738\ldots = 3.544 \)

Area \( \triangle AOB = \frac{1}{2}(1.5)(1.5)\sin 2.738\ldots \)

= 0.4409…

Area major sector \( AOB = \frac{1}{2}1.5(3.544\ldots) = 3.987\ldots \)

Cross-sectional area of milk = 0.4409…+3.987… = 4.428…

∴ Volume of milk = 4.428… × 3 = 13.3 m³

Exercise 8B

1 a \( \sin 144° = \sin 36° \)

b \( \cos 210° = -\cos 30° \)

c \( \tan 230° = \tan 30° \)

d \( \sin \left( \frac{7\pi}{8} \right) = \sin \left( \frac{\pi}{8} \right) \)

e \( \tan \left( \frac{7\pi}{3} \right) = \tan \left( \frac{\pi}{3} \right) \)

f \( \cos \left( \frac{2\pi}{6} \right) = -\cos \left( \frac{\pi}{6} \right) \)

2 \( \cos \theta = -\frac{12}{13} \therefore \tan \theta = -\frac{5}{12} \)

3 \( \sec \theta = -\frac{5}{4} \therefore \cos \theta = -\frac{4}{5} \)

\( \tan \theta = \frac{3}{4} \therefore \sin \theta = -\frac{3}{5} \)

4 a \( 2 + 4\cos \theta \)

i max value = 6 when \( \theta = 2\pi \)

ii min value = –2 when \( \theta = \pi \)

b \( 5 - 3\sin \theta \)

i max value = 8 when \( \theta = \frac{3\pi}{2} \)

ii min value = 2 when \( \theta = \frac{\pi}{2} \)

c \( 2\sin \theta - 1 \)

i max value = 1 when \( \theta = \frac{\pi}{2} \)

ii min value = –3 when \( \theta = \frac{3\pi}{2} \)

d \( -2\cos \theta - 3 \)

i max value = –1 when \( \theta = \pi \)

ii min value = –5 when \( \theta = 2\pi \)
Investigation – trigonometric identities

1. a. \( \sin \theta = \cos(90 - \theta), \quad \cos \theta = \sin(90 - \theta), \)
   \[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\cos(90 - \theta)}{\sin(90 - \theta)} \]
   
   b. \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)
   
   c. \( \sin^2 \theta + \cos^2 \theta = 1 \)
   
   d. \( \tan^2 \theta + 1 = \sec^2 \theta \)
   
   e. \( \cot^2 \theta + 1 = \csc^2 \theta \)

Investigation – exact values of \( \sin, \cos \) and \( \tan \)

1. a. \( \frac{\pi}{4} = \frac{\sqrt{2}}{2} \)
   \( \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \tan \frac{\pi}{4} = 1 \)

   b. \( \frac{\pi}{3} = \frac{\sqrt{3}}{2} \)
   \( \cos \frac{\pi}{3} = \frac{1}{2} \quad \tan \frac{\pi}{3} = \sqrt{3} \)

   c. \( \frac{\pi}{6} = \frac{1}{2} \)
   \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \)

Exercise 8C

1. a. \( \sin \theta = \frac{1}{4} \quad \frac{\pi}{4} \leq \theta \leq \pi \)
   \( \cos^2 \theta = 1 - \sin^2 \theta = 1 - \left( \frac{1}{16} \right) = \frac{15}{16} \quad : \quad \cos \theta = -\frac{\sqrt{15}}{4} \)
   \( \tan \theta = \frac{\sin \theta}{\cos \theta} = -\frac{1}{\sqrt{15}} \)

2. \( \cos \theta = -\frac{12}{13} \quad 0 \leq \theta \leq \pi \) (\( \theta \) lies in 2nd quadrant)
   \( \sin^2 \theta = 1 - \cos^2 \theta = 1 - \left( -\frac{12}{13} \right)^2 = \frac{25}{169} \quad : \quad \sin \theta = \frac{5}{13} \)
   \( \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{5}{12} \)

3. \( \sin \left( \arcsin \left( \frac{\sqrt{3}}{2} \right) \right) = \arctan \left( \frac{\pi}{3} \right) \)
   \( \sin \left( \frac{\pi}{3} \right) - \arctan \left( \frac{\pi}{6} \right) \)
   \( = \frac{\pi}{6} = \frac{1}{2} \) (QED)

4. \( \sin \theta \quad \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin^2 \theta + (1 - \cos \theta)}{\sin \theta} \)
   \( = \frac{2 - 2 \cos \theta}{\sin \theta} = \frac{2(1 - \cos \theta)}{\sin \theta} = \frac{2}{\sin \theta} \) (QED)

5. \( \tan \theta + \cot \theta = \frac{\sin \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = \sec \theta \csc \theta \) (QED)

6. \( \cot^2 \theta - \cos^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta} - \cos^2 \theta = \frac{\cos^2 \theta - \cos^2 \theta \sin^2 \theta}{\sin^2 \theta} = \frac{\cos \theta (1 - \sin^2 \theta)}{\sin^2 \theta} = \cos \theta \csc^2 \theta \) (QED)

Exercise 8D

1. a. \( \sin 75^\circ = \sin (30^\circ + 45^\circ) \)
   \( = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \)
   \( = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \)
   \( = \frac{\sqrt{3} + 1}{2\sqrt{2}} \)

   b. \( \tan 15^\circ = \tan (60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} \)
   \( = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \)

   c. \( \sec 105^\circ = \frac{1}{\cos (60^\circ + 45^\circ)} = \frac{1}{\frac{1}{2} \sqrt{3} - \frac{1}{2} \sqrt{3}} = \frac{2\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} = -\frac{2\sqrt{3}}{3} \)

2. a. \( \cos 70^\circ \cos 10^\circ + \sin 70^\circ \sin 10^\circ = \cos 60^\circ = \frac{1}{2} \)

   b. \( \tan 75^\circ = \tan (30^\circ + 45^\circ) \)
   \( = \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ} \)
   \( = \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \frac{1 + \sqrt{3}}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{1 + \sqrt{3}}{4} \)

3. \( \sin \theta = \frac{24}{25} \quad 0 < \theta < \frac{\pi}{2} \)
   \( \cos^2 \theta = 1 - \left( \frac{24}{25} \right)^2 = \frac{49}{625} \quad : \quad \cos \theta = \frac{7}{25} \quad \tan \theta = \frac{24}{7} \)

   a. \( \sin \phi = \frac{3}{5} \quad \frac{2}{5} < \phi < \pi \Rightarrow \cos \phi = -\frac{4}{5} \quad \tan \phi = -\frac{3}{4} \)

   b. \( \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{24}{7} \times \frac{3}{4} = \frac{75}{28} \)

   c. \( \cot(A + B) = \frac{1}{\tan(A + B)} = \frac{1 - \tan A \tan B}{\tan A + \tan B} \times \frac{1}{tan A + tan B} \)
   \( = \frac{1}{\tan A \tan B} + 1 \cot A \cot B - 1 = \cot A \cot B - 1 \) (QED)

4. \( \sin(A + B) = \sin A \cos B + \cos A \sin B \)
   \( = \frac{\sin A}{\cos B} + \frac{\sin B}{\cos A} = \tan A + \tan B \) (QED)

5. \( \sin (A + B) = \sin A \cos B + \cos A \sin B \quad \sin (A + B) = \frac{\sin A}{\cos B} + \frac{\sin B}{\cos A} \)
   \( = \tan A + \tan B \) (QED)

b. \( (\sin A + \cos A)(\sin B + \cos B) \)
   \( = \sin A \sin B + \sin A \cos B + \cos A \sin B + \cos A \cos B \)
   \( = (\sin A \cos B + \cos A \sin B) + (\sin A \sin B + \cos A \cos B) \)
   \( = \sin (A + B) + \cos (A - B) \) (QED)
6  a Let \( \alpha = \arctan \left( \frac{1}{4} \right) \) and \( \beta = \arctan \left( \frac{3}{5} \right) \)

Then \( \tan \alpha = \frac{1}{4} \) and \( \tan \beta = \frac{3}{5} \)

\[ \therefore \tan(\alpha + \beta) = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \]

\[ = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \cdot \frac{3}{5}} = \frac{17}{20} = 1 \]

\[ \therefore \alpha + \beta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \]

But \( 0 < \alpha < \frac{\pi}{4} \) or \( \frac{5\pi}{4} \), so \( 0 < \alpha + \beta < \frac{5\pi}{4} \)

\[ \therefore \alpha + \beta = \frac{\pi}{4} \text{ i.e. } \arctan \left( \frac{1}{4} \right) + \arctan \left( \frac{3}{5} \right) = \frac{\pi}{4} \quad \text{(QED)} \]

b Let \( \arctan(4) = \gamma \)

Then \( \tan \gamma = 4 = \frac{1}{\tan \alpha} \)

\[ \therefore \gamma = \frac{\pi}{2} - 2 \]

Similarly if \( \delta = \arctan \left( \frac{5}{3} \right) \), then \( \delta = \frac{\pi}{2} - \beta \)

\[ \therefore \arctan(4) + \arctan \left( \frac{5}{3} \right) = \frac{\pi}{2} - \alpha + \frac{\pi}{2} - \beta \]

\[ = \pi - (\alpha + \beta) \]

\[ = \pi - \frac{\pi}{4} \]

\[ = \frac{3\pi}{4} \]

Exercise 8E

1  a \( \sin(A + B) = \sin A \cos B + \cos A \sin B \)

Let \( A = B = \theta \)

\[ \sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta \]

\[ \therefore \sin 2\theta = 2 \sin \theta \cos \theta \quad \text{(QED)} \]

b \( \cos(A + B) = \cos A \cos B - \sin A \sin B \)

Let \( A = B = \theta \)

\[ \cos 2\theta = \cos \theta \cos \theta - \sin \theta \sin \theta \]

\[ = \cos^2 \theta - \sin^2 \theta \quad \text{(QED)} \]

c \( \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \)

Let \( A = B = \theta \), \( \tan 2\theta = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} \)

\[ \therefore \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} \quad \text{(QED)} \]

2 \( \cos \alpha = \frac{4}{5} \)

\( \sin \alpha = \frac{3}{5} \)

\( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \)

\[ = \frac{4}{5} \times \frac{7}{25} + \frac{3}{5} \times \frac{24}{25} \]

\[ = \frac{100}{125} \text{ or } - \frac{44}{125} = \frac{4}{5} \text{ or } - \frac{44}{125} \]

3 \( \cos A = \frac{1}{3} \), \( \cos 2A = \cos^2 A - 1 = 2\left( \frac{1}{3} \right)^2 - 1 = \frac{-7}{9} \)

\( \cos 4A = 2\cos^2 2A - 1 = 2\left( \frac{-7}{9} \right)^2 - 1 = \frac{17}{81} \)

Exercise 8F

1  a \[ \theta \]

\[ \tan \left( \theta + \frac{\pi}{3} \right) \tan \left( \theta - \frac{\pi}{3} \right) = \left( \frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} \right) \left( \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} \right) \]

\[ = \frac{\tan^2 \theta - 3}{1 - 3\tan^2 \theta} \quad \text{(QED)} \]

b \( \cos^2 A + \sin^2 A = \cos^2 A + 1 = \frac{1}{2}(\cos 2A + 1) + 1 \]

\[ = \frac{1}{2}(\cos 2A + 3) \]

c \( \sin^4 A = (\sin^2 A)^2 = \frac{1}{4}(1 - \cos 2A)^2 \)

6  a \( (1 + \tan^2 \theta)(1 - \cos 2\theta) = (1 + \tan^2 \theta)(1 - (1 - 2\sin^2 \theta)) \]

\[ = (1 + \tan^2 \theta) 2\sin^2 \theta \]

\[ = \sec^2 \theta 2\sin^2 \theta \]

\[ = 2 \sin^2 \theta \quad \text{(QED)} \]

b \( (1 + \tan^2 \theta)(1 + \cos 2\theta) = \sec^2 \theta(1 + 2\cos^2 \theta - 1) \]

\[ = \frac{1}{\cos \theta}(2\cos^2 \theta) \]

\[ = 2 \quad \text{(QED)} \]

7  a \( \frac{1 - \cos 2A}{1 + \cos 2A} = \frac{1 - (1 - 2\sin^2 A)}{1 + (2\cos^2 A - 1)} \)

\[ = \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A \quad \text{(QED)} \]

b \( \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{\frac{\sin^2 A}{\cos^2 A}}{\frac{\sin^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} \)

\[ = \frac{\cos^2 A}{\sin^2 A} = \cot A \quad \text{(QED)} \]

c \( \frac{\sin 2A}{1 - \cos 2A} = \frac{2\sin A \cos A}{2\cos^2 A - 1} \)

\[ = \frac{\sin A \cos A}{\sin^2 A} \]

\[ = \cot A \quad \text{(QED)} \]

d \( \cos 3A = \cos(A + 2A) \)

\[ = \cos A \cos 2A - \sin A \sin 2A \]

\[ = \cos A(2\cos^2 A - 1) - \sin A(2\sin A \cos A) \]

\[ = 2\cos^3 A - \cos A - 2\sin^2 A \cos A \]

\[ = 2\cos^3 A - \cos A - 2\cos A \]

\[ = 2\cos^3 A - \cos A \]

\[ = 4\cos^3 A - 3\cos A \quad \text{(QED)} \]

Exercise 8G
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1 i \[ f(x) = 7 \sin \left( 6 \left(x - \frac{\pi}{12}\right) \right) + 3 \]

\[ \text{a} \quad \text{amplitude} = 7 \quad \text{period} = \frac{2\pi}{6} = \frac{\pi}{3} \]

\[ \text{phase shift} = \frac{\pi}{12} \]

\[ \text{b} \quad \text{min. value} = -4, \text{max. value} = 10 \]
\[ f(x) = -3\sin\left(2x + \frac{\pi}{2}\right) - 5 = -3\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 5 \]

- amplitude = 3
- period = \(\frac{2\pi}{2} = \pi\)
- phase shift = \(\frac{\pi}{4}\)
- min value = -8
- max value = -2

2 \[ V(t) = 220\sin(120\pi t) \]

- max = 220
- min = -220
- amplitude = 220
- period = \(\frac{2\pi}{120\pi} = \frac{1}{60}\)

3 \[ h(t) = a\sin\left[b(t + c)\right] + d \]

- \(a = \frac{14.4 - 1.2}{2} = 6.6\)
- \(d = \frac{14.4 + 1.2}{2} = 7.8\)
- \(\frac{2\pi}{b} = 12 \quad \therefore b = \frac{\pi}{6}\)
- \(h(t) = 6.6\sin\left[\frac{\pi}{6}(t + c)\right] + 7.8\)
- \(h(2.85) = 14.4\)
- \(\therefore 14.4 = 6.6\sin\left[\frac{\pi}{6}(8.25 + c)\right] + 7.8\)
- \(\therefore \sin\left[\frac{\pi}{6}(8.25 + c)\right] = 1 \quad \therefore \frac{\pi}{6}(8.25 + c) = \frac{\pi}{2}\)
- \(\therefore 8.25 + c = 3 \quad \therefore c = -5.25\)
- \(a = 6.6 \quad b = \frac{\pi}{6} \quad c = -5.25 \quad d = 7.8\)

4 \[ f(x) = a\sin[b(x + c)] + d \]

- \(a = \frac{12.75 - 10.65}{2} = 1.05\)
- \(d = \frac{12.75 + 10.65}{2} = 11.7\)
- \(\frac{2\pi}{b} = 365 \quad \therefore b = \frac{2\pi}{365}\)
- \(f(x) = 1.05\sin\left[\frac{2\pi}{365}(x + c)\right] + 11.7\)

On June 21, \(x = 172\), \(f(172) = 12.75\)
- \(12.75 = 1.05\sin\left[\frac{2\pi}{365}(172 + c)\right] + 11.7\)
- \(\sin\left[\frac{2\pi}{365}(172 + c)\right] = 1 \quad \therefore \frac{2\pi}{365}(172 + c) = \frac{\pi}{2}\)
- \(\therefore 172 + c = \frac{365}{4} \quad \therefore c = -80.75\)
- \(a = 1.05 \quad b = \frac{2\pi}{365} \quad c = -80.75 \quad d = 11.7\)

On July 4, \(x = 185\), \(f(185) = 12.72\)
- \(\therefore 12.7\) hours

Exercise 8H

1 \[ \cos\left(\arcsin\left(\frac{\sqrt{2}}{2}\right)\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\]
2 a \( \sin\left(\arcsin\frac{1}{2} + \arccos\frac{1}{2}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \sin\frac{5\pi}{6} = 1 \)

b Let \( \arcsin \frac{3}{5} = \theta \Rightarrow \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5} \)
\[ \arccos\left(-\frac{4}{5}\right) = \phi \Rightarrow \cos \phi = -\frac{4}{5}, \sin \phi = \frac{3}{5} \]
\[ \cos\left(\arcsin \frac{3}{5} - \arccos\left(-\frac{4}{5}\right)\right) = \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi \]
\[ = \frac{4}{5} \times \frac{4}{5} - \frac{3}{5} \times \frac{3}{5} = \frac{7}{25} \]

c Let \( \tan\left(\frac{3}{4}\right) = \phi \), \( \tan \phi = \frac{3}{4} \)
\[ \tan\left(2\tan\left(\frac{3}{4}\right)\right) = \tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta} \]
\[ = \frac{3}{7} \times \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \frac{16}{7} \]

3 a Let \( \arcsin a = \theta \), \( \sin \theta = a \cos \theta = \sqrt{1 - a^2} \)
\[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{a}{\sqrt{1 - a^2}} \]
\[ \therefore \tan(\arcsin a) = \frac{a}{\sqrt{1 - a^2}} \quad (\text{QED}) \]

b Let \( \sin a = \theta \), \( \arcsin \theta \cos \theta = a \)
\[ \cos(\sin a + \arcsin \theta) = \cos(\theta + \phi) \]
\[ = \cos \theta \cos \phi - \sin \theta \sin \phi \]
\[ = \sqrt{1 - a^2}(a) - a\sqrt{1 - a^2} \]
\[ = 0 \quad (\text{QED}) \]

c Let \( \arccos \theta = a \), \( \cos \theta = a \sin \theta = \sqrt{1 - a^2} \)
\[ \tan(\arccos a) = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{1 - a^2}}{a} \quad (\text{QED}) \]

Exercise 8I

1 \( 3\sin x = 2\tan x \quad -\pi \leq x \leq \pi \)
\( 3\sin x = \frac{2\sin x}{\cos x} \)
\( 3\sin x \cos x - 2\sin x = 0 \)
\( \sin(3\cos x - 2) = 0 \)
\( \sin x = 0 \) or \( \cos x = \frac{2}{3} \)
\( x = 0, \pm \pi \) or \( x = \pm 0.841 \)

2 \( \cot \theta + \sin \theta = 6 \quad [0, \pi] \)
\[ \frac{\cos \theta}{\sin \theta} + \sin \theta = 6 \]
Using GDC: \( \theta = 0.170 \)

3 \( 3\cos^2 \theta = 2\cos^2 \theta \quad [-\pi, \pi] \)
\( 3(2\cos^2 \theta - 1) = 2\cos^2 \theta \)
\( 4\cos^2 \theta = 3 \) \( \cos^2 \theta = \frac{3}{4} \), \( \cos \theta = \pm \frac{\sqrt{3}}{2} \)
\[ \theta = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6} \]

4 \( 3\tan^2 \theta - \frac{14}{\cos \theta} + 18 = 0 \)
\( 3(\sec^2 \theta - 1) - 14\sec \theta + 18 = 0 \)
\( 3\sec^2 \theta - 14\sec \theta + 15 = 0 \)
\( (3\sec \theta - 3)(\sec \theta - 3) = 0 \)
\( \sec \theta = \frac{3}{3} \) or 3

5 \( \sin x - \cos x = 1 \quad 0 \leq x \leq \pi \)
\( 2\sin x \cos x = 1 \) \( \cos x = \frac{1}{\sqrt{2}} \)
\( 2\sin x \cos x - 2\cos^2 x = 0 \)
\( 2\cos x = 0 \) or \( \sin x = \cos \frac{x}{2} \Rightarrow \tan \frac{x}{2} = 1 \)
\( \frac{x}{2} = \frac{\pi}{4} \) or \( x = \frac{\pi}{4} \)
\[ \therefore x = \frac{\pi}{2} \text{ or } \pi \]

6 \( \cos \theta + \sin \theta = 2 \quad -\pi \leq \theta \leq \pi \)
\[ \frac{1}{\sin \theta} + \sin \theta = 2 \]
\( 1 + \sin^2 \theta = 2\sin \theta \)
\( \sin^2 \theta - 2\sin \theta + 1 = 0 \)
\( (\sin \theta - 1)^2 = 0 \)
\( \sin \theta = 1 \therefore \theta = \frac{\pi}{2} \)

7 \( \frac{\sin x - 3\cos x}{\sin x - \cos x} = 7 \)
\( \sin x - 3\cos x = 7\sin x - 7\cos x \)
\( 4\cos x = 6\sin x \)
\[ \therefore \tan x = \frac{4}{6} = \frac{2}{3} \]
\[ \therefore x = \frac{\pi}{6} \]

8 \( \frac{x}{2}\sin 2x = \sqrt{x} \sin x \quad 0 \leq x \leq 2\pi \)
From graph, \( x = 0, \pi, 5.17, 2\pi \)

9 \( -5x^3 \cos 8x = \tan x \quad 0 \leq x \leq \frac{\pi}{2} \)
Using GDC, \( x = 0, 0.294, 0.536, 1.02, 1.32 \)
Exercise 8J

1 a  \( QR^2 = 5^2 + 8^2 - 2 \times 5 \times 8 \cos 30^\circ = 19.17 \ldots \)

\( QR = 4.44 \)

\( \cos Q = \frac{5^2 + 4.44^2 - 8^2}{2 \times 5 \times 4.44} = -0.4342 \ldots \)

\( \hat{PQR} = 116^\circ \) \( \hat{PQ} = 34.3^\circ \)

b  \( \hat{XZ}^2 + 4^2 + 5^2 - 2 \times 4 \times 5 \cos 95^\circ = 44.48 \ldots \)

\( \hat{XZ} = 6.67 \)

\( \cos \hat{Z} = \frac{4^2 + 6.67^2 - 5^2}{2 \times 4 \times 6.67} = 0.6650 \ldots \)

\( \hat{XZY} = 48.3^\circ \) \( \hat{YZX} = 36.7^\circ \)

c  \( \cos A = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4} = 0.859375 \)

\( \hat{BAC} = 30.8^\circ \)

\( \cos C = \frac{8^2 + 5^2 - 4^2}{2 \times 8 \times 5} = 0.9125 \)

\( \hat{ACB} = 24.1^\circ \) \( \hat{ABC} = 125^\circ \)

\[ \frac{\sin Q}{15^\circ} = \frac{150}{70} \Rightarrow \text{sin} Q = 0.1380 \]

\( Q = 7.934^\circ \)

\( R = 180^\circ - (15^\circ + 7.93^\circ) = 157.07^\circ \)

\( \frac{PQ}{\sin 157.07^\circ} = \frac{150}{\sin 15^\circ} \Rightarrow \text{PQ} = 225.84 \text{ km} \)

extra distance travelled = 230 – 225.84 = 4.16... km

time lost = \( \frac{4.16...}{400} \) hours = 0.0104... hours = 37 sec (nearest second)

Exercise 8K

1 a  \( \hat{ACB} = 180^\circ - (30^\circ + 125^\circ) = 25^\circ \)

\( \hat{ACB} = 25^\circ \)

\( \frac{AC}{\sin 125^\circ} = \frac{10}{\sin 30^\circ} \Rightarrow \text{AC} = 16.4 \text{ cm} \)

\( \frac{AB}{\sin 25^\circ} = \frac{10}{\sin 30^\circ} \Rightarrow \text{AB} = 8.45 \text{ cm} \)

b  \( \hat{PQR} = 95^\circ \)

\( \frac{RP}{\sin 95^\circ} = \frac{7}{\sin 45^\circ} \Rightarrow \text{RP} = 9.86 \text{ cm} \)

\( \frac{QR}{\sin 95^\circ} = \frac{7}{\sin 45^\circ} \Rightarrow \text{QR} = 6.36 \text{ cm} \)

Exercise 8L

1  \( PR^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \cos 125^\circ = 418.129... \)

\( PR = 20.448... \)

Area = \( \frac{1}{2} \times 10 \times 13 \sin 125^\circ + \frac{1}{2} \times 15 \times 20.448 \sin 70^\circ \)

= 197 sq.units
\[ \sqrt{3}\sin \theta + \cos \theta = 1 \quad 0 \leq \theta \leq 2\pi \]
\[ \frac{2\sqrt{3}\sin \theta + 1 - r^2}{1 + r^2} = 1 \]
\[ 2\sqrt{3}\sin \theta + 1 - r^2 = 1 + r^2 \]
\[ 2r^2 - 2\sqrt{3}\sin \theta = 0 \]
\[ 2r(t - \sqrt{3}) = 0 \]
\[ \tan \frac{\theta}{2} = 0 \text{ or } \tan \frac{\theta}{2} = \sqrt{3} \]
\[ \theta = 0, \frac{2\pi}{3} \text{ or } 2\pi \]

2 a \quad \sin 165^\circ = \sin(120^\circ + 45^\circ) = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}

b \quad \tan 105^\circ = \tan(60^\circ + 45^\circ) = \frac{2\tan 60^\circ + 2\tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}}

c \quad \cos \frac{5\pi}{12} = \cos \left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \sqrt{2} = \frac{\sqrt{6} - \sqrt{2}}{4}

d \quad \tan \frac{\pi}{8} = \frac{2\tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}} = \frac{-2 + \sqrt{2}}{2} = -1 + \sqrt{2} \quad \text{since } \tan \frac{\pi}{8} > 0 \quad \therefore \tan \frac{\pi}{8} = \sqrt{2} - 1

a \quad \frac{1}{1 - \tan \theta} = \frac{1}{1 - \sin \theta \cos \theta} = \frac{\cos \theta}{\cos \theta - \sin \theta}
\quad \therefore \frac{\cos \theta}{\cos \theta - \sin \theta} = \frac{1}{1 - \tan \theta} \quad (QED)

b \quad \cos(A - B) = \frac{\cos A \cos B + \sin A \sin B}{\cos B \cos A}
\quad = \frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B}
\quad = 1 + \tan A \tan B \quad (QED)

c \quad \cos 3A = \cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A
\quad = \cos A(2\cos^2 A - 1) - \sin A 2\sin A \cos A
\quad = 2\cos^2 A - \cos A - 2\sin^2 \cos A

\sin 3A = \sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A
\quad = \sin A(2\cos^2 A - 1) + \cos 2A \sin A \cos A
\quad = 2\sin A \cos^2 A - \sin A + 2 \sin A \cos^2 A
\quad = 4 \sin A \cos^2 A - \sin A
cos 3A − sin 3A = 2 cos^3 A − cos A − 2 sin^2 A cos A
− 4 sin A cos^2 A + sin A
= 2 cos A (1 − sin^2 A) − cos A − 2 sin^2 A cos A
− 4 sin A cos^2 A + sin A
= cos A − 4 sin^2 A + A sin A
= cos A (1 − sin^2 A) + sin A (1 − sin^2 A)
= (cos A + sin A) (1 − sin^2 A) (QED)

\[ \begin{align*}
\sin 2\theta (1 - 2\sin^2 \theta) &= 2 \sin 2\theta \\
&= 4 \theta \quad (QED)
\end{align*} \]

\[ \begin{align*}
1 + 2 \cos 2A + \cos 4A &= 1 + 2 \cos 2A + 2 \cos^2 2A - 1 \\
&= 2 \cos 2A (1 + \cos 2A) \\
&= 2 \cos 2A (1 + 2 \cos^2 A - 1) \\
&= 4 \cos^2 A \cos 2A \quad (QED)
\end{align*} \]

\[ \begin{align*}
\text{Let } \arcsin \theta &= 0, \sin \theta = \frac{1}{\sqrt{3}}, \cos \theta = \frac{\sqrt{2}}{\sqrt{3}} \\
\arccos \frac{1}{2} &= \phi, \cos \phi = \frac{1}{2}, \sin \phi = \frac{\sqrt{3}}{2} \\
\cos \left( \arcsin \frac{1}{\sqrt{3}} - \arccos \frac{1}{2} \right) &= \cos (\theta - \phi) \\
&= \cos \theta \cos \phi + \sin \theta \sin \phi \\
&= \frac{2}{\sqrt{2}} \times \frac{1}{2} + \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{4 + 3 \sqrt{2}}{10}
\end{align*} \]

\[ \begin{align*}
\text{Let } \arccos \left( -\frac{3}{5} \right) &= \theta \quad \text{and } \sin \theta = -\frac{3}{5}, \quad \cos \theta = \frac{4}{5} \\
\sin \left[ 2 \arccos \left( -\frac{3}{5} \right) \right] &= \sin (2\theta) = 2 \sin \theta \cos \theta \\
&= 2 \left( \frac{4}{5} \right) \left( -\frac{3}{5} \right) = \frac{-24}{25}
\end{align*} \]

\[ \begin{align*}
\text{arctan} (-1) &= -\frac{\pi}{4} \\
\text{Let } \arccos \left( -\frac{4}{5} \right) &= \theta \quad \text{and } \cos \theta = -\frac{4}{5}, \quad \sin \theta = \frac{3}{5} \\
\sin \left[ \text{arctan} (-1) + \arccos \left( -\frac{4}{5} \right) \right] &= \sin \left( \theta - \frac{\pi}{4} \right) \\
&= \sin \theta \cos \frac{\pi}{4} - \cos \theta \sin \frac{\pi}{4} \\
&= \frac{3}{5} \times \frac{\sqrt{2}}{2} - \frac{4}{5} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{10}
\end{align*} \]

\[ \begin{align*}
\text{Let } \arcsin x = \theta, \sin \theta = x, \cos \theta = \sqrt{1 - x^2} \\
\arccos x &= \phi, \cos \phi = x, \sin \phi = \sqrt{1 - x^2} \\
\sin \left[ \arcsin x - \arccos x \right] &= \sin (\theta - \phi) \\
&= \sin \theta \cos \phi - \cos \theta \sin \phi \\
&= x^2 - (\sqrt{1 - x^2})^2 = x^2 - (1 - x^2) = 2x^2 - 1 \quad (QED)
\end{align*} \]

\[ \begin{align*}
\tan (2x + y) &= \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y} \\
\tan \left( \frac{\pi}{4} \right) &= \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y} = 1 \\
\therefore \tan 2x + \tan y &= 1 - \tan 2x \tan y \\
\tan y(1 + \tan 2x) &= 1 - \tan 2x \\
\tan y &= \frac{1 - \tan 2x}{1 + \tan 2x} \\
\tan y &= \frac{\frac{1 - \tan^2 x}{1 + \tan^2 x}}{1 - \tan x} = 1 - \tan x + 2 \tan x \\
\therefore \tan y &= 1 - \frac{2 \tan x - \tan^2 x}{1 + 2 \tan x - \tan^2 x} \quad (QED)
\end{align*} \]

**Review exercise**

1. \( \cos (A - B) - \cos (A + B) = \cos A \cos B + \sin A \sin B - (\cos A \cos B - \sin A \sin B) = 2 \sin A \sin B \quad (QED) \)

\[ \begin{align*}
\sin 3x \sin x &= -1 \\
\text{Let } A &= 3x, B &= x \\
\cos 2x - \cos 4x &= 2 \sin 3x \sin x \\
\cos 2x - \cos 4x &= -2 \\
\cos 2x - (2 \cos^2 2x - 1) &= -2 \\
2 \cos^2 2x - 1 &= -2 \\
2 \cos^2 2x &= 0 \\
(2 \cos 2x - 3)(\cos 2x + 1) &= 0 \\
\cos 2x &= -1 \\
2x &= \pi \\
x &= \frac{\pi}{2}
\end{align*} \]

2. \[ \begin{align*}
a \quad &\sin y + \sin x = 1.1 \Rightarrow y = \arcsin (1.1 - \sin x) \\
\cos y + \sin 2x = 1.8 \Rightarrow y = \arcsin (1.8 - \sin 2x) \\
b \quad &\text{Using GDC, } x = 0.619, \quad y = 0.546 \\
&\text{or } x = 1.09, \quad y = 0.216
\end{align*} \]
3 a  \[ \hat{A}DB = 110^\circ \quad \hat{A}BO = 180^\circ - (15^\circ + 110^\circ) = 55^\circ \]
\[ \hat{O}BC = \frac{1}{2}(180^\circ - 70^\circ) = 55^\circ \]
\[ \therefore \hat{A}BC = 55^\circ + 55^\circ = 110^\circ \]

b  In \( \Delta ABC \), \[ \frac{AB}{\sin 55^\circ} = \frac{0.6}{\sin 15^\circ} \quad \therefore AB = 1.90 \text{ m} \]

c  ![Diagram of triangle ABC]
\[ AX = 1.898 - 0.3 = 1.598... \]
\[ \tan 15^\circ = \frac{OX}{1.598...} \]
\[ \therefore OX = 0.428 \text{ m} \]
\[ \text{radius} = 0.428 \text{ m} \]

4 a  \[ f(x) = \frac{\sin x + 3\cos x}{4 + 3\cos x}, \quad 0 \leq x \leq 2\pi \]
For vertical asymptotes,
\[ 4 + 3\cos x = 0 \]
\[ \cos x = -\frac{4}{3} \text{ (no solution)} \]
\[ \therefore \text{no vertical asymptotes} \quad \text{(QED)} \]

b  \[ f(0) = \frac{2}{7}, \quad (0, \frac{2}{7}) \]

c  \( p = 3.87 \quad q = 5.55 \)

d  ![Graph of function]

e  Points of intersection at \( x = 0.510, 3.53, 3.99, 5.49 \)
\( f(x) > g(x) \) for \( 0.510 < x < 3.53 \)
and \( 3.99 < x < 5.49 \)

f  Max. value of \( f(x) - g(x) \) is 2.39
(when \( x = 1.88 \))

5 a  ![Diagram of triangle ABC]
\[ DN^2 = 5^2 + 7^2 \]
\[ DN = \sqrt{74} = 8.60 \text{ cm} \]

6 a  Area \( \Delta ABC = \frac{1}{2}ab \sin C \)
\( (h = \text{length of perpendicular from } C \text{ to } AB) \)
or area \( \Delta ABC = \frac{1}{2}ab \sin C \)
\[ \therefore \frac{1}{2}h = \frac{1}{2}ab \sin C \]
\[ \therefore h = \frac{ab}{c} \sin C \quad \text{(QED)} \]
b  In \( \triangle BCD \), \( \tan 30^\circ = \frac{10}{BC} \) \(: \ BC = 17.3 \ \text{m} \)

In \( \triangle ACD \), \( \tan 45^\circ = \frac{10}{AC} \) \(: \ AC = 10 \ \text{m} \)

In \( \triangle ABC \), \( AB^2 = 17.3^2 + 10^2 - 2 \times 17.3 \times 10 \cos 150^\circ = 700 \)

\( AB = 26.5 \ \text{m} \)

From \( a, h = \frac{ab \sin C}{c} = \frac{17.3 \times 10}{26.5} \sin 150^\circ \)

\( h = 3.27 \ \text{m} \)

In \( \triangle CDE \), \( CD = 7.2^2 - 3^2 = 42.84 \),

\( CD = \sqrt{42.84} \)

In \( \triangle ABC \), \( BC^2 = 16.8^2 - 7^2 = 233.24 \),

\( BC = \sqrt{233.24} \)

\( \cos \theta = \frac{7}{16.8} \) \(: \ \theta = 1.141 \)

Major arc of large circle = \( 7(2\pi - 2\theta) = 28.008... \)

Major arc of small circle = \( 3(2\pi - 2\theta) = 12.003... \)

Length of belt = \( 2BC + 2CD + 28.008 + 12.003 \)

= 83.6 cm