Complete Solutions to Miscellaneous Exercises 5

1. We are given the transformation \[ T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 \\ 2x_1 - 3x_2 \end{bmatrix} \]. How do we show that this is linear?

Need to show that both \( T(u + v) = T(u) + T(v) \) and \( T(ku) = kT(u) \), where \( k \) is a scalar, are satisfied. This is definition (6.2).

Let \( u = \begin{bmatrix} a \\ b \end{bmatrix} \) and \( v = \begin{bmatrix} c \\ d \end{bmatrix} \). Then by applying the given transformation we have

\[ T(u) = T\left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a + 5b \\ 2a - 3b \end{bmatrix} \]
\[ T(v) = T\left( \begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} c + 5d \\ 2c - 3d \end{bmatrix} \]

Checking \( T(u + v) = T(u) + T(v) \):

\[ T(u + v) = T\left( \begin{bmatrix} a + c \\ b + d \end{bmatrix} \right) = \begin{bmatrix} (a + c) + 5(b + d) \\ 2(a + c) - 3(b + d) \end{bmatrix} = T(u) + T(v) \]

because \( T\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 5x_2 \\ 2x_1 - 3x_2 \end{bmatrix} \).

Thus we have \( T(u + v) = T(u) + T(v) \).

Checking \( T(ku) = kT(u) \):

\[ T(ku) = T\left( \begin{bmatrix} ka \\ kb \end{bmatrix} \right) = \begin{bmatrix} ka + 5kb \\ 2ka - 3kb \end{bmatrix} = kT(u) \]

This means we have \( T(ku) = kT(u) \).

Hence the given transformation is linear because both conditions \( T(u + v) = T(u) + T(v) \) and \( T(ku) = kT(u) \) are satisfied.
2. By taking the transpose we have

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix}$$

**How do we show T is a linear map (transformation)?**

By Definition (6.2) we need to show both the following conditions:

$$T(u + v) = T(u) + T(v)$$ and $$T(ku) = kT(u)$$ where \( k \) is a scalar

Let \( u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) and \( v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \). Then by applying the given transformation we have

$$T(u) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix}$$ and $$T(v) = T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} 2y_1 - 3y_2 + 4y_3 \\ -y_1 + y_2 \end{bmatrix}$$

We have

$$T(u + v) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1 + y_1) - 3(x_2 + y_2) + 4(x_3 + y_3) \\ -(x_1 + y_1) + (x_2 + y_2) \end{bmatrix}$$

Applying the given linear map:

$$= \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 - 3y_2 + 4y_3 \\ -y_1 + y_2 \end{bmatrix}$$

$$= T(u) + T(v)$$

Let \( k \) be a scalar. For \( T \) to be linear we also need to show $$T(ku) = kT(u)$$:

$$T(ku) = T\left(\begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}\right) = \begin{bmatrix} 2kx_1 - 3kx_2 + 4kx_3 \\ -kx_1 + kx_2 \end{bmatrix} = kT(u)$$

Hence \( T \) is a linear map.

The standard matrix \( S \) is given by the coefficients of \( x_1, x_2 \) and \( x_3 \).
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3. An example of a non-linear transformation \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is

\[
f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ xy \end{bmatrix}
\]

This does not satisfy \( f(ku) = kf(u) \) where \( u = \begin{bmatrix} x \\ y \end{bmatrix} \) because

\[
f(ku) = f \begin{bmatrix} kx \\ ky \end{bmatrix} = \begin{bmatrix} k^2x^2 \\ k^2xy \end{bmatrix} = k^2f \begin{bmatrix} x \\ y \end{bmatrix} = k^2f(u)
\]

We have \( f(ku) = k^2f(u) \neq kf(u) \) which means that \( f \) is not a linear transformation.

4. (a) The standard matrix for \( T(x_1, x_2, x_3) = (3x_2 + 2x_3, 3x_1 - 4x_2) \) is determined by evaluating \( T(e_1), T(e_2) \) and \( T(e_3) \) where \( e_1 = (1, 0, 0)^T \), \( e_2 = (0, 1, 0)^T \) and \( e_3 = (0, 0, 1)^T \) [or reading off the coefficients of \( x_1, x_2 \) and \( x_3 \)]:

\[
T(e_1) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T(e_2) = T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \text{and} \quad T(e_3) = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]

Thus the standard matrix \( S \) is given by

\[
S = \begin{pmatrix} 0 & 3 & 2 \\ 3 & -4 & 0 \end{pmatrix}
\]

(b) By Proposition (6.15) \( T \) is one-to-one \( \iff \) \( \ker(T) = \{0\} \). How can we find \( \ker(T) \)?

By finding \( x_1, x_2 \) and \( x_3 \) such that \( T(x_1, x_2, x_3) = (3x_2 + 2x_3, 3x_1 - 4x_2) = \mathbf{0} \):

\[3x_2 + 2x_3 = 0 \quad (*)\]
\[3x_1 - 4x_2 = 0 \quad (**)\]

From \((*)\) we have \( 3x_2 = -2x_3 \) \( \implies \) \( x_2 = -\frac{2}{3}x_3 \). Let \( x_3 = 3t \) where \( t \) is any real number.

Then \( x_2 = -2t \). Substituting \( x_2 = -2t \) into the bottom equation \((**\)) we have

\[3x_1 - 4(-2t) = 0 \quad \implies \quad x_1 = \frac{8}{3}t\]
We have non-zero solutions because \( x_1 = -\frac{8}{3}t, \ x_2 = -2t \) and \( x_3 = 3t \) where \( t \in \mathbb{R} \). This means that \( \ker(T) = \left\{ \begin{pmatrix} -8t/3 \\ -2t \\ 3t \end{pmatrix} \right\} \neq 0 \).

Hence \( T \) is not one-to-one.

(c) From part (b) we have \( \dim(\ker(T)) = 1 \) because there is only one free variable \( (t) \). Using the dimension theorem (6.12) which says:

\[
\dim(\ker(T)) + \dim(\text{range}(T)) = n \quad (†)
\]

where \( n \) is the dimension of the domain. In this case the domain is \( \mathbb{R}^3 \) because we are given \( T(x_1, \ x_2, \ x_3) = (3x_2 + 2x_1, \ 3x_1 - 4x_2) \) which means \( T: \mathbb{R}^3 \to \mathbb{R}^2 \).

What is the dimension of \( \mathbb{R}^3 \)?

It is 3. Substituting \( n = 3 \) and \( \dim(\ker(T)) = 1 \) into (†) gives

\[
1 + \dim(\text{range}(T)) = 3 \quad \Rightarrow \quad \dim(\text{range}(T)) = 2
\]

This means that \( \text{range}(T) = \mathbb{R}^2 \) and since we have \( T: \mathbb{R}^3 \to \mathbb{R}^2 \) therefore \( T \) is onto.

5. (a) How do we check that the given set \( S \) is a subspace of \( \mathbb{R}^2 \)?

By using Proposition (4.7). A nonempty subset \( S \) is a subspace of a vector space \( V \iff \)

\( (a) \quad \mathbf{0} \in S \) [Zero vector is in \( S \)].

\( (b) \quad \text{If} \ \mathbf{u} \text{ and } \mathbf{v} \text{ are vectors in } S \text{ then any linear combination } k\mathbf{u} + c\mathbf{v} \text{ is also in } S. \)

Clearly the zero vector \( \mathbf{0} \) is in \( S \) because \( \mathbf{0} \cdot \mathbf{u} = 0 \).

Let \( \mathbf{v} \) and \( \mathbf{w} \) be vectors in \( S \) and \( k \) and \( c \) be scalars. Need to show that \( k\mathbf{v} + c\mathbf{w} \) is also in \( S \) for \( S \) to be a subspace of \( \mathbb{R}^2 \):

\[
(k\mathbf{v} + c\mathbf{w}) \cdot \mathbf{u} = k(\mathbf{v} \cdot \mathbf{u}) + c(\mathbf{w} \cdot \mathbf{u})
\]

\[
= k(0) + c(0) = 0 \quad [\mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} = 0 \text{ because } \mathbf{v}, \ \mathbf{w} \in S]
\]

Since both conditions of Proposition (4.7) are satisfied therefore \( S \) is a subspace of \( \mathbb{R}^2 \).

(b) (i) We are given

\[
T\left[ \begin{array}{c} 1 \\ -1 \end{array} \right] = \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] \text{ and } T\left[ \begin{array}{c} 0 \\ -1 \end{array} \right] = \left[ \begin{array}{c} 1 \\ 3 \end{array} \right]
\]

We have to write \( \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) in terms of \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ -1 \end{pmatrix} \):

\[
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{gives } a = 1, \ b = -1
\]

\[
\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{gives } c = 0, \ d = -1
\]

We can find \( T(\mathbf{e}_1) \) and \( T(\mathbf{e}_2) \) by using the above:
\[ T(e_1) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T \begin{bmatrix} 1 \\ -1 \end{bmatrix} = T \begin{bmatrix} 1 \\ -1 \end{bmatrix} - T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = T \begin{bmatrix} 2 \\ 1 \end{bmatrix} - T \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \]  

Substituting \( a = 1 \) and \( b = -1 \)

\[ T(e_2) = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = T \begin{bmatrix} 0 \\ -1 \end{bmatrix} - T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = T \begin{bmatrix} 0 \\ 2 \end{bmatrix} - T \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \]  

Substituting \( c = 0 \) and \( d = -1 \)

(ii) The matrix representation \( A \) with respect to the standard basis is

\[ A = \begin{bmatrix} 1 & -2 \\ -3 & -3 \end{bmatrix} \]

where \( T \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} \)

(iii) Let \( w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \) then

\[ T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \]

\[ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 8 \end{bmatrix} \]

Taking the inverse matrix

\[ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{-3 - 2} \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \frac{5}{-10} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \]

That is \( w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \).

6. (a) The given transformation \( S \) is

\[ S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 4x_2 + 2x_3 \\ 2x_1 + 7x_2 - x_3 \\ -x_1 - 8x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix} \]

Remember the standard matrix is given by the coefficients of \( x_1, x_2 \) and \( x_3 \). This means that the standard matrix, call it \( A \), of operator \( S \) is

\[ A = \begin{bmatrix} 1 & -4 & 2 \\ 2 & 7 & -1 \\ -1 & -8 & 2 \\ 2 & 1 & 1 \end{bmatrix} \]

(b) To find a basis for the range of \( T \) we take the transpose of matrix \( A \) and then place this into row echelon form:
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\[ A^T = \begin{pmatrix} 1 & -4 & 2 \\ 2 & 7 & -1 \\ -1 & -8 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 2 \\ -4 & 7 & -8 & 1 \\ 2 & -1 & 2 & 1 \end{pmatrix} \]

Labelling the rows of \( A^T \):

\[
R_1 \begin{pmatrix} 1 & 2 & -1 & 2 \end{pmatrix} \\
R_2 \begin{pmatrix} -4 & 7 & -8 & 1 \end{pmatrix} \\
R_3 \begin{pmatrix} 2 & -1 & 2 & 1 \end{pmatrix}
\]

Carrying out the row operations \( R_2 + 4R_1 \) and \( R_3 - 2R_1 \):

\[
R_1 \begin{pmatrix} 1 & 2 & -1 & 2 \end{pmatrix} \\
R_2' = R_2 + 4R_1 \begin{pmatrix} 0 & 15 & -12 & 9 \end{pmatrix} \\
R_3' = R_3 - 2R_1 \begin{pmatrix} 0 & -5 & 4 & -3 \end{pmatrix}
\]

Executing \( R_2' \div 3 \):

\[
R_1 \begin{pmatrix} 1 & 2 & -1 & 2 \end{pmatrix} \\
R_2^* = R_2' \div 3 \begin{pmatrix} 0 & 5 & -4 & 3 \end{pmatrix} \\
R_3' \begin{pmatrix} 0 & -5 & 4 & -3 \end{pmatrix}
\]

Executing \( R_3' + R_2^* \):

\[
R_1 \begin{pmatrix} 1 & 2 & -1 & 2 \end{pmatrix} \\
R_2^* \begin{pmatrix} 0 & 5 & -4 & 3 \end{pmatrix} \\
R_3^* = R_3' + R_2^* \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}
\]

A basis for the range are the non-zero rows of the last matrix, that is

\[
\begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -4 \\ 3 \end{pmatrix}
\]

7. (a) \( T: V \to W \) is a linear transformation if both the following conditions are satisfied:

\[ T(u + v) = T(u) + T(v) \quad \text{and} \quad T(ku) = kT(u) \]

for all vectors \( u \) and \( v \) in \( V \) and any scalar \( k \).

(b) The kernel of \( T \), \( \ker T \), is the set of vectors \( v \) in \( V \) of \( T: V \to W \) such that

\[ T(v) = 0. \]

(c) We need to prove that:

\( T \) is injective, this means one to one, \( \iff \ker T = \{0\} \)

\textit{Proof}.

(\( \Rightarrow \)). We assume \( T \) is one to one. By Proposition (6.3) we have \( T(0) = 0 \). Since \( T \) is one to one therefore there can be \textbf{no} other vector in \( V \) which is transformed to the zero vector under \( T \). Hence \( \ker T = \{0\} \).

(\( \Leftarrow \)). We assume \( \ker T = \{0\} \). What do we need to prove?

\( T \) is one to one (injective). \textit{How?}
By Definition (6.13) which says:
Transformation $T$ is one to one $\iff u \neq v$ implies $T(u) \neq T(v)$.

Let $u$ and $v$ be in $V$ such that $u \neq v$. We have

$$T(u - v) = T(u) - T(v) \neq O$$

because $T$ is linear.

Because if $T(u) - T(v) = O$ then we have

$$T(u - v) = T(u) - T(v) = O \Rightarrow u - v = O \Rightarrow u = v$$

This is contradiction because we had $u \neq v$ therefore

$$T(u) - T(v) \neq O$$

Hence $T(u) \neq T(v)$. By Definition (6.13) we conclude that $T$ is one to one (injective).

(d) We need to find $\ker T$ which means we need to find $x$, $y$ and $z$ such that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ x + y + z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{(*)}$$

We have to solve the simultaneous equations

$$x + y = 0 \quad \text{(*)}$$
$$x + y + z = 0 \quad \text{(**)}$$

From the top equation (*) we have $x = -y$. Let $y = t$ where $t$ is any real number then $x = -t$. Substituting these, $x = -t$ and $y = t$, into the bottom equation (**) gives $z = 0$. Thus $\ker T$ is given by

$$\ker T = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{where } t \in \mathbb{R}$$

We conclude that $\ker(T) = \text{span}\{(-1, 1, 0)^T\}$.

8. a. To find a basis for the image (range) of $T$ we need to transpose the given matrix and we can call this new matrix $A$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = A$$

How do we find a basis for the image of $T$?

It is the non-zero rows of the (reduced) row echelon form of matrix $A$:

$$R_1 \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$
$$R_2 \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$
$$R_3 \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$$
$$R_4 \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

Carrying out the row operations $R_2 - R_1$ and $R_3 - R_5$:
\[ R_1 \quad \begin{pmatrix} 1 & 1 & 0 \\ \end{pmatrix} \]
\[ R_2^* = R_2 - R_1 \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]
\[ R_3^* = R_3 - R_1 \quad \begin{pmatrix} 0 & -2 & 2 \\ \end{pmatrix} \]
\[ R_4 \quad \begin{pmatrix} 0 & 1 & 1 \\ \end{pmatrix} \]

Carrying out the row operation \( R_3^* + 2R_4 \):
\[ R_4 \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]
\[ R_2^* = R_2 \quad \begin{pmatrix} 0 & 1 & 1 \\ \end{pmatrix} \]
\[ R_4' = R_3^* + 2R_4 \quad \begin{pmatrix} 0 & 0 & 4 \\ \end{pmatrix} \]
\[ R_4 = R_2^* \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]

Dividing the third row by 4 and interchanging rows \( R_2^* \) and \( R_4 \):
\[ R_4 = R_4 \quad \begin{pmatrix} 1 & 1 & 0 \\ \end{pmatrix} \]
\[ R_2^* = R_4 \quad \begin{pmatrix} 0 & 1 & 1 \\ \end{pmatrix} \]
\[ R_4'' = R_3^* / 4 \quad \begin{pmatrix} 0 & 0 & 1 \\ \end{pmatrix} \]
\[ R_4 = R_2^* \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]

Executing \( R_2' - R_4'' \):
\[ R_1 \quad \begin{pmatrix} 1 & 1 & 0 \\ \end{pmatrix} \]
\[ R_2'' = R_2' - R_3^* \quad \begin{pmatrix} 0 & 1 & 0 \\ \end{pmatrix} \]
\[ R_3^* \quad \begin{pmatrix} 0 & 0 & 1 \\ \end{pmatrix} \]
\[ R_4 \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]

Finally executing \( R_4 - R_2'' \) gives us a matrix in reduced row echelon form:
\[ R_1' = R_4 - R_2'' \quad \begin{pmatrix} 1 & 0 & 0 \\ \end{pmatrix} \]
\[ R_2^* \quad \begin{pmatrix} 0 & 1 & 0 \\ \end{pmatrix} \]
\[ R_3^* \quad \begin{pmatrix} 0 & 0 & 1 \\ \end{pmatrix} \]
\[ R_4 \quad \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} \]

A basis \( B \) for the image of \( T \) are the non-zero rows of this last matrix, that is
\[ B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

The columns of the matrix associated with \( T \) does span \( \mathbb{F}^3 \) because a basis for this is the set \( B \) above which is the standard basis for \( \mathbb{F}^3 \).

b. The transformation \( T \) is onto because the basis for the image of \( T \) is the set \( B \) given in part a which is the standard basis for \( \mathbb{F}^3 \) which means that the range of \( T \) is \( \mathbb{F}^3 \) and we are given \( T : \mathbb{F}^4 \rightarrow \mathbb{F}^3 \).

c. A basis for the null space of \( T \) can be found by placing the given matrix into reduced row echelon form and solving the resulting equations \( Rx = O \):
\[ R_1 \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ \end{pmatrix} \]
\[ R_2 \quad \begin{pmatrix} 1 & 1 & -1 & 1 \\ \end{pmatrix} \]
\[ R_3 \quad \begin{pmatrix} 0 & 0 & 2 & 1 \\ \end{pmatrix} \]

Carrying out the row operation \( R_2 - R_1 \):
\[
R_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}
\]

\[
R_2^* = R_2 - R_1
\]

\[
R_3
\]

Executing \( R_3 + R_2^* \):

\[
R_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}
\]

Dividing the bottom row by 2 gives

\[
R_1
\]

\[
R_2^* = R_2^* / 2
\]

Executing \( R_2^* - R_3^* \):

\[
R_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Dividing the middle row by \(-2\) gives

\[
R_1
\]

\[
R_2^" = R_2^*/(-2)
\]

Carrying out the row operation \( R_1 - R_2^" \) gives us the reduced row echelon form matrix \( \mathbf{R} \):

\[
\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Null space is found by solving \( \mathbf{R}x = \mathbf{0} \) which is

\[
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

which gives \( x_1 = -x_2 \), \( x_3 = x_4 = 0 \)

A basis \( B' \) for the null space is \( B' = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \).

d. By part c we have one vector in the basis of the null space so the dimension of the null space is 1 which means that \( \text{nullity}(T) = 1 \).

Since \( \text{nullity}(T) = 1 \) therefore the given transformation \( T \) is not one to one because Proposition (6.17) in the main text says:

\( T \) is one to one \( \Leftrightarrow \text{nullity}(T) = 0 \).
9. We are given that
\[ T\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \]

However we need to find \( T\begin{bmatrix} 7 \\ -11 \end{bmatrix} \). How?

We need to write \( T\begin{bmatrix} 7 \\ -11 \end{bmatrix} \) in terms of \( T\begin{bmatrix} 3 \\ -5 \end{bmatrix} \) and \( T\begin{bmatrix} -1 \\ 2 \end{bmatrix} \). Let \( a \) and \( b \) be the scalars such that
\[ a\begin{bmatrix} 3 \\ -5 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \end{bmatrix} \]

Writing these out as equations and solving
\[ \begin{cases} 3a - b = 7 \\ -5a + 2b = -11 \end{cases} \]
gives \( a = 3 \) and \( b = 2 \)

Thus we have
\[ T\begin{bmatrix} 7 \\ -11 \end{bmatrix} = 3T\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 2T\begin{bmatrix} -1 \\ 2 \end{bmatrix} \]
[Because \( T \) is linear]
\[ = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \]
\[ = \begin{bmatrix} 3 + 6 \\ -3 + 0 \\ 6 - 4 \end{bmatrix} \]
\[ = \begin{bmatrix} 9 \\ -3 \\ 2 \end{bmatrix} \]

Thus \( T\begin{bmatrix} 7 \\ -11 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 2 \end{bmatrix} \).

10. One definition of mathematics is the science of patterns. What pattern do you notice about the given vector \( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \)?

Notice that
\[ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \]

Since \( L \) is a linear transformation therefore
11. (a) We are given that matrix $A$ is of size $6 \times 5$. What does this mean?

Means the matrix $A$ has 6 rows and 5 columns. We are also given $T(x) = Ax$ which means we have

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 2 \\
1 & 1 & 1 & 4 & 4
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
-1 \\
-1 \\
-1 \\
2 \\
4
\end{bmatrix}
$$

(b) Since the range of $T$ is a subset of $\mathbb{R}^6$, therefore the maximum number of linearly independent vectors in the range is 6.

(c) We are given that nullity of $A$ is 0. By the dimension theorem (6.12) we have

$$
\text{nullity}(T) + \text{rank}(T) = n
$$

Substituting $\text{nullity}(T) = 0$ and $n = 5$ gives $\text{rank}(T) = 5$. Since $\text{rank}(T) = 5$ therefore range of $T$ cannot equal $\mathbb{R}^6$ because $\dim(\mathbb{R}^6) = 6$ so $T$ is not onto.

(d) Since $\text{nullity}(T) = 0$ therefore $T$ is one-to-one because Proposition (6.17) says:

$$
T: V \to W \text{ is one to one } \iff \text{nullity}(T) = 0
$$

12. Let $A$ be the matrix representing the transformation $T: \mathbb{F}^2 \to \mathbb{F}^3$. We have

$$
A = \begin{bmatrix}
T(e_1) & T(e_2)
\end{bmatrix}
$$

We are given

$$
T(e_1) = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix} \quad \text{and} \quad T(e_2) = \begin{bmatrix} 3 \\ k \\ 0 \end{bmatrix}
$$

Thus the matrix $A$ is given by $A = \begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix}$. For what values of $h$ and $k$ is the transformation one-to-one?

By Proposition (6.15) we have $T$ is one to one $\iff \ker(T) = \mathbf{0}$ where $\mathbf{0}$ is the zero vector. How do we find the kernel of $T$?

Let $x = \begin{bmatrix} x \\ y \end{bmatrix}$ then $\ker(T) = \{x \mid Ax = \mathbf{0}\}$. For one to one we need $x = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$ which means the only solution to $Ax = \mathbf{0}$ is $x = 0$ and $y = 0$.

By expanding $Ax = \mathbf{0}$ we have

$$
Ax = \begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ x+ky \\ hx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$
From the bottom row we have \(hx = 0 \Rightarrow h \neq 0\)

Why do we need \(h \neq 0\)?

Because if \(h = 0\) then \(x \neq 0\) will satisfy the bottom row. Remember the only solution to the above equations is \(x = y = 0\).

The first two rows are the simultaneous equations

\[
\begin{align*}
2x + 3y &= 0 \quad (1) \\
x + ky &= 0 \quad (2)
\end{align*}
\]

Multiplying equation (2) by 2 and subtracting from (1) we have

\[
\begin{align*}
2x + 3y &= 0 \\
-2x + 2ky &= 0 \\
(3 - 2k)y &= 0
\end{align*}
\]

We have \(3 - 2k \neq 0 \Rightarrow k \neq \frac{3}{2}\).

This means that the given transformation is one to one for all real values of \(h\) and \(k\) provided \(h \neq 0\) and \(k \neq \frac{3}{2}\).

13. (a) The zero vector in \(C[0,1]\) is the constant function 0.

(b) Let \(f, g \in C^1[0,1]\). What do we need to show?

Required to show both the following conditions:

\[
T(f + g) = T(f) + T(g) \quad \text{and} \quad T(kf) = kT(f)
\]

where \(k\) is a scalar.

Checking \(T(f + g) = T(f) + T(g)\):

\[
T(f + g) = \frac{d}{dx}(f(x) - 2(f + g)(x))
\]

\[
= \frac{d}{dx}[(f(x) - 2f(x)) - 2g(x)]
\]

\[
= \frac{d}{dx}(f(x)) + \frac{d}{dx}g(x) - 2f(x) - 2g(x)
\]

\[
= \frac{df}{dx}(x) - 2f(x) + \frac{dg}{dx}(x) - 2g(x)
\]

\[
= T(f) + T(g)
\]

Checking \(T(kf) = kT(f)\):

\[
T(kf) = \frac{d}{dx}(kf(x) - 2kf(x))
\]

\[
= k \frac{d}{dx}[(f(x) - 2f(x)]
\]

\[
= k \left[ \frac{df}{dx}(x) - 2f(x) \right] = kT(f)
\]

Hence \(T\) is a linear transformation.
(c) The kernel of $T$ is the set of functions $f \in C^1[0, 1]$ which satisfies

$$\frac{df}{dx}(x) - 2f(x) = 0$$

Rearranging this we have

$$\frac{d}{dx}[f(x)] = 2f(x)$$

$$\frac{d[f(x)]}{f(x)} = 2dx$$

Integrating both sides gives

$$\int \frac{d[f(x)]}{f(x)} = 2 \int dx$$

$$\ln(f(x)) = 2x + C$$

Taking exponentials of both sides gives

$$f(x) = e^{2x+C} = e^{2x}e^C = Ae^{2x}$$

where $A = e^C$ is a constant.

We have $\ker(T) = Ae^{2x}$ which means $\ker(T) = \text{span}\{e^x\}$. Therefore there is only one vector in basis of $\ker(T)$ which implies $\dim(\ker(T)) = 1$.

14. We are given that $T : V \rightarrow W$ is a linear transformation.

(a) We need to show that $\ker(T)$ is a subspace of $V$.

How do we show this result?

By using Proposition (4.7) which says:

A non-empty subset $S$ in a vector space $V$ is a subspace of $V$ if and only if:

(a) $O \in S$

(b) $u, v \in S$ then for any scalars $k, c$ we have $ku + cv \in S$

We need to show both the conditions, (a) and (b), of Proposition (4.7).

Proof.

Checking condition (a):

Since $T(O) = O$ therefore $O \in \ker(T)$ so condition (a) is satisfied. This also means that $\ker(T)$ is a non-empty subset of $V$.

Checking condition (b):

Let $u$ and $v$ be vectors in $\ker(T)$ and $k, c$ be any scalars. Consider the transformation of the linear combination $ku + cv$:

$$T(ku + cv) = kT(u) + cT(v)$$

[Because $T$ is linear]

$$= kO + cO$$

[Because $u, v \in \ker(T)$ so $T(u) = T(v) = O$]

$$= O$$

This means that $ku + cv \in \ker(T)$.

Thus both conditions of (4.7) are satisfied so we conclude $\ker(T)$ is a subspace of $V$.

(b) We need to show that $\text{im}(T)$ is a subspace of $W$. How?
Again we use Proposition (4.7) as described in part (a) above. Remember image of $T$ is the same as the range of $T$.

**Proof.**

Checking condition (a):
Since $T(O) = O$ therefore $O \in im(T)$ so condition (a) is satisfied. This also means that $im(T)$ is a non-empty subset of $W$.

Checking condition (b):
Let $u$ and $v$ be vectors in $im(T)$ and $k$, $c$ be any scalars. We need to show $ku + cv$ is also in $im(T)$.

Since $u$, $v \in im(T)$ therefore there must be vectors $u'$, $v' \in V$ such that $T(u') = u$ and $T(v') = v$. We have

$$ku + cv = kT(u') + cT(v') = T(ku + cv')$$

Because $T$ is linear.

We know $u'$, $v' \in V$ and $V$ is a vector space so therefore $ku' + cv' \in V$. This means that $ku + cv \in im(T)$.

Both the conditions of (4.7) are satisfied so we conclude that $im(T)$ is a subspace of $W$.

(c) *How do we find the dimensions of the kernel and image (range) of $T$?*

We can use the dimension theorem (6.12) which says:

$$\text{dim}(\text{range}(T)) + \text{dim}(\text{ker}(T)) = n$$

where $n$ is the dimension of $V$ if we have $T: V \rightarrow W$.

In our case we have $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ so $n=3$. We need to find the dimensions of either image of $T$ or the kernel of $T$. Let us find the dimensions of kernel which is the solution set of $T(x, y, z) = O$:

$$x + 2y - z = 0 \quad (1)$$
$$y + z = 0 \quad (2)$$
$$x + y - 2z = 0 \quad (3)$$

From the middle equation (2) we have $y = -z$. Let $z = t$ where $t$ is any real number. Then $y = -t$. Substituting these $y = -t$ and $z = t$ into the top equation (1) gives

$$x - 2t - t = 0 \quad \text{gives} \quad x = 3t$$

Our solution set is the kernel of $T$ which is given by

$$\text{ker}(T) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

where $t \in \mathbb{R}$.

Since $\text{ker}(T) = \text{span}\{(3, -1, 1)^T\}$ therefore $\text{dim}(\text{ker}(T)) = 1$.

Substituting $\text{dim}(\text{ker}(T)) = 1$ and $n = 3$ into the above equation (*) gives

$$\text{dim}(\text{range}(T)) + 1 = 3 \quad \Rightarrow \quad \text{dim}(\text{range}(T)) = 2$$
Hence we have \( \dim(\ker(T)) = 1 \) and \( \dim(\text{range}(T)) = 2. \) Remember range and image are identical terms so \( \dim(\text{image}(T)) = 2. \)

15. We are given that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is linearly independent and we need to prove that \( S = \{v_1, v_2, \ldots, v_n\} \) is linearly independent. How?

Required to prove that \( k_1v_1 + k_2v_2 + \cdots + k_nv_n = \mathbf{0} \implies k_1 = k_2 = \cdots = k_n = 0. \)

**Proof.**

Consider the linear combination
\[ k_1v_1 + k_2v_2 + \cdots + k_nv_n = \mathbf{0} \]
where \( k_1, k_2, \ldots, k_n \) are scalars. Taking the transformation of this gives
\[ T(k_1v_1 + k_2v_2 + \cdots + k_nv_n) = T(\mathbf{0}) \]

Since \( T \) is linear we have
\[ k_1T(v_1) + k_2T(v_2) + \cdots + k_nT(v_n) = T(\mathbf{0}) = \mathbf{0} \]

We are given that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is linearly independent therefore
\[ k_1 = k_2 = \cdots = k_n = 0 \]

We have \( k_1v_1 + k_2v_2 + \cdots + k_nv_n = \mathbf{0} \implies k_1 = k_2 = \cdots = k_n = 0 \) which means that \( S = \{v_1, v_2, \ldots, v_n\} \) is linearly independent.

We need to show that \( S = \{v_1, v_2, \ldots, v_n\} \) is linearly independent \( \implies \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is linearly independent.

For example consider the linear transformation \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x-y \end{pmatrix} \]

Let \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \), then these are linearly independent but

\[ T(v_1) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \]

Since \( T(v_1) = \mathbf{0} \) therefore \( T(v_1) \) and \( T(v_2) \) are linearly dependent because one of the vectors is the zero vector.

16. (a) A mapping \( T: P_2 \to P_1 \) is a linear mapping if both the following conditions are satisfied:

\[ T(p + q) = T(p) + T(q) \text{ and } T(kp) = kT(p) \]

where \( p \) and \( q \) are any vectors in \( P_2 \) and \( k \) is a scalar.

**Checking** \( T(kp) = kT(p) \):

Let \( k \) be a scalar and \( p = a_0 + a_1t + a_2t^2 \) then applying
\[ T\left(a_0 + a_1t + a_2t^2 \right) = 3a_1 + 2a_2t + a_0t^2 + (a_1 + a_2)t^3 \]
gives
(b) The matrix $A$ is given by $A = \begin{bmatrix} [T(1)]_C & [T(t)]_C & [T(t^2)]_C \end{bmatrix}$. We need to find $T(1)$, $T(t)$ and $T(t^2)$ where $T(a_0 + a_1 t + a_2 t^2) = 3a_1 + 2a_2 t + a_2 t^2 + (a_1 + a_2) t^3$:

$T(1) = T(1 + 0t + 0t^2) = 3(0) + 2(0) t + 0t^2 + (0 + 0) t^3 = t^2$

$T(t) = T(0 + 1t + 0t^2) = 3(1) + 2(0) t + 0t^2 + (1 + 0) t^3 = 3 + t^3$

$T(t^2) = T(0 + 0t + 1t^2) = 3(0) + 2(1) t + 0t^2 + (0 + 1) t^3 = 2t + t^3$

We need to write each of these vectors $T(1)$, $T(t)$ and $T(t^2)$ as coordinates of the basis $C = \{1, t, t^2, t^3\}$:

$T(1) = t^2 = 0(1) + 0(t) + 1(t^2) + 0(t^3)$

$T(t) = 3 + t^3 = 3(1) + 0(t) + 0(t^2) + 1(t^3)$

$T(t^2) = 2t + t^3 = 0(1) + 2(t) + 0(t^2) + 1(t^3)$

What is $A = \begin{bmatrix} [T(1)]_C & [T(t)]_C & [T(t^2)]_C \end{bmatrix}$ equal to?

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(c) By using the above matrix $A$ we need to determine $T(2 + 5t - t^2)$. What are the coordinates of $2 + 5t - t^2$ with respect to the basis $B = \{1, t, t^2\}$?

$$\begin{bmatrix} 2 + 5t - t^2 \end{bmatrix}_B = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

We have $A \begin{bmatrix} 2 + 5t - t^2 \end{bmatrix}_B = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -2 \\ 2 \\ 4 \end{pmatrix}$. This means that

$$T(2 + 5t - t^2) = 15 - 2t + 2t^2 + 4t^3$$

Check: Applying $T(a_0 + a_1 t + a_2 t^2) = 3a_1 + 2a_2 t + a_2 t^2 + (a_1 + a_2) t^3$:
\[ T(2+5t-t^2) = 3(5) + 2(-1)t + 2t^2 + (5-1)t^3 = 15 - 2t + 2t^2 + 4t^3 \]

(d) Yes we can do the same with \( T(p) := t^3 + p(t) \) but we will need to replace the matrix \( A \) above with a new matrix.

17. (a) Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \) and \( m \) be any scalar. Then \( T: \mathbb{R}^n \to \mathbb{R}^k \) is a linear transformation if

\[ T(u + v) = T(u) + T(v) \quad \text{and} \quad T(mu) = mT(u) \quad (m \text{ is a scalar}) \]

(b) (i) Let \( u = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) and \( v = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \) then applying the given transformation

\[ T([x, y, z]) = [x + 2y, 2y - 3z, z + x, x] \]

we have

\[ T(u) = T(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = \begin{bmatrix} a + 2b \\ 2b - 3c \\ c + a \\ a \end{bmatrix} \quad \text{and} \quad T(v) = T(\begin{bmatrix} d \\ e \\ f \end{bmatrix}) = \begin{bmatrix} d + 2e \\ 2e - 3f \\ f + d \\ d \end{bmatrix} \]

Evaluating the other way we have

\[ T(u + v) = T(\begin{bmatrix} a + d \\ b + e \\ c + f \end{bmatrix}) = \begin{bmatrix} (a + d) + 2(b + e) \\ 2(b + e) - 3(c + f) \\ (c + f) + (a + d) \\ a + d \end{bmatrix} \]

\[ = \begin{bmatrix} (a + 2b) + (d + 2e) \\ 2b - 3c + (2e - 3f) \\ c + a + (f + d) \\ a + d \end{bmatrix} = T(u) + T(v) \]

Hence we have \( T(u + v) = T(u) + T(v) \). Let us check the second condition, \( T(mu) = mT(u) \):
Thus the given transformation $T$ is linear.

(ii) The standard matrix $S$ of the given linear transformation

$T([x, y, z]) = [x + 2y, 2y - 3z, z + x, x]$ is determined by reading off the coefficients of $x, y$ and $z$:

$$S = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Because $T = \begin{bmatrix} x + 2y \\ 2y - 3z \\ z + x \\ x \end{bmatrix}$

(c) We need to prove that $T(v)(v \in \mathbb{R}^n)$ is uniquely determined by the vectors $T(b_1), T(b_2), \ldots, T(b_n)$ where $\beta = \{b_1, b_2, \ldots, b_n\}$ is a basis of $\mathbb{R}^n$.

Proof:

Let $v \in \mathbb{R}^n$ be an arbitrary vector. Since $\beta = \{b_1, b_2, \ldots, b_n\}$ is a basis of $\mathbb{R}^n$ therefore we can write the vector $v$ uniquely as

$$v = k_1b_1 + k_2b_2 + \cdots + k_nb_n$$

where the $k$’s are scalars. Taking the transformation of this we have

$$T(v) = T(k_1b_1 + k_2b_2 + \cdots + k_nb_n) = k_1T(b_1) + k_2T(b_2) + \cdots + k_nT(b_n) \quad [\text{Because } T \text{ is linear}]$$

Thus $T(v)$ can be expressed uniquely by the vectors $T(b_1), T(b_2), \cdots, T(b_n)$.

18. (a) Need to show that $\beta = \{1 + x, 1 - x, e^x, (1 + x)e^x\}$ is linearly independent.

Let $k_1, k_2, k_3$ and $k_4$ be scalars such that

$$k_1(1 + x) + k_2(1 - x) + k_3e^x + k_4(1 + x)e^x = 0$$

Expanding this out gives

$$(k_1 + k_2)(1 + x) + (k_3 + k_4)e^x + k_4xe^x = 0$$

Equating coefficients of $xe^x$ gives $k_4 = 0$.

Equating coefficients of $e^x$ gives $k_3 + k_4 = 0 \implies k_3 = 0$ \ [Because $k_4 = 0$].

Equating coefficients of $x$ gives $k_1 - k_2 = 0 \implies k_1 = k_2$.

Equating constants gives $k_1 + k_2 = 0 \implies k_1 = -k_2$.

From the last two lines we have $k_1 = k_2 = 0$. Hence all our scalars $k_1 = k_2 = k_3 = k_4 = 0$.

What does this mean?
Means that the vectors in $\beta = \{1 + x, \ 1 - x, \ e^x, \ (1 + x)e^x\}$ are linearly independent.

Since these vectors span the given subspace $V$ therefore they are a basis for $V$.

(b) Using the expanded version from part (a) which is:

$$ (k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = 1 - xe^x \quad (\bigstar) $$

Equating coefficients of $xe^x$ in $(\bigstar)$ gives $k_4 = -1$.

Equating coefficients $e^x$ gives $k_3 + k_4 = 0$. Since $k_4 = -1$ therefore $k_3 = 1$.

Equating coefficients of $x$ and constants of $(\bigstar)$ gives the simultaneous equations

$$ \begin{cases} k_1 - k_2 = 0, \\ k_1 + k_2 = 1 \end{cases} \Rightarrow k_1 = k_2 = \frac{1}{2} $$

The coordinates of the given vector $u = 1 - xe^x$ with respect to the basis $\beta$ is

$$ [u]_\beta^\beta = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ -1 \end{bmatrix} $$

(c) We need to differentiate each of the terms in $\beta = \{1 + x, \ 1 - x, \ e^x, \ (1 + x)e^x\}$:

$$ \frac{d}{dx}(1 + x) = 1, \quad \frac{d}{dx}(1 - x) = -1, \quad \frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(1 + x)e^x = 2e^x + xe^x $$

Writing each of these $1, \ -1, \ e^x, \ 2e^x + xe^x$ in terms of the basis vectors $\beta$ we have:

$$(k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = 1 \text{ implies } k_1 = k_2 = \frac{1}{2}, \quad k_3 = k_4 = 0$$

$$(k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = -1 \text{ implies } k_1 = k_2 = -\frac{1}{2}, \quad k_3 = k_4 = 0$$

$$(k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = e^x \text{ implies } k_1 = k_2 = k_3 = k_4 = 0, \quad k_3 = 1$$

$$(k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = 2e^x + xe^x \text{ implies } k_1 = k_2 = 0, \quad k_3 = k_4 = 1$$

We have

$$ [1]^\beta = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \quad [-1]^\beta = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}, \quad [e^x]^\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [2e^x + xe^x]^\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} $$

The matrix representation of this differentiation operator is given by

$$ [D]_\beta^\beta = \begin{bmatrix} [1]^\beta & [-1]^\beta & [e^x]^\beta & [2e^x + xe^x]^\beta \end{bmatrix} $$

The matrix is

$$ [D]_\beta^\beta = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(d) We first differentiate the given $u = 1 - xe^x$:

$$ \frac{d}{dx}[1 - xe^x] = -(e^x + xe^x) = -e^x - xe^x $$
\[ [Du]^\beta \] is the coordinates of \( Du \) with respect to the basis \( \beta \). This means we need to find the scalars \( k_1, k_2, k_3 \) and \( k_4 \) such that
\[
(k_1 + k_2) + (k_1 - k_2)x + (k_3 + k_4)e^x + k_4xe^x = -e^x - xe^x
\]
which gives \( k_1 = k_2 = 0, k_4 = -1 \) and \( k_3 = 0 \). Thus \( [Du]^\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \).

Working out \( [D]_\beta^\beta[u]^\beta = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \) From part (b)

Hence we have \( [Du]^\beta = [D]_\beta^\beta[u]^\beta \).

19. (a) i. Let \( u = \begin{bmatrix} a \\ b \end{bmatrix} \) and \( v = \begin{bmatrix} c \\ d \end{bmatrix} \) be our vectors in \( \mathbb{R}^2 \). How do we show whether the given map \( f \) is linear or not?

Need to show that both \( f(u + v) = f(u) + f(v) \) and \( f(ku) = kf(u) \), where \( k \) is a scalar, are satisfied. This is definition (6.2).

We are given \( f : \mathbb{R}^2 \rightarrow P_5 : (a, b) \rightarrow (a + b)x^5 \).

Checking \( f(u + v) = f(u) + f(v) \):
\[
f(u + v) = f\left( \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right) = f\left( \begin{bmatrix} a + c \\ b + d \end{bmatrix} \right) = [(a + c) + (b + d)]x^5 = (a + b)x^5 + (c + d)x^5 = f(u) + f(v)
\]

Checking \( f(ku) = kf(u) \):
\[
f(ku) = f\left( k \begin{bmatrix} a \\ b \end{bmatrix} \right) = f\left( \begin{bmatrix} ka \\ kb \end{bmatrix} \right) = (ka + kb)x^5
\]
Thus since \( f \) satisfies both \( f(u + v) = f(u) + f(v) \) and \( f(ku) = kf(u) \) therefore \( f \) is a linear map.
ii. We need to check whether \( f : M(2, 2) \to \mathbb{R}^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (ad, bc) \) is linear or not. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( f(A) = \begin{pmatrix} ad \\ bc \end{pmatrix} \).

Checking \( f(kA) = kf(A) \). It is easier to check this condition first.

\[
\begin{align*}
f(kA) &= f(k \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\
&= f(\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}) \\
&= \begin{pmatrix} k\cdot ad \\ k\cdot bc \end{pmatrix} = k^2 f(A) = k f(A) \quad [\text{Not Equal}]
\end{align*}
\]

Hence the given map is **not** linear because \( f(kA) \neq kf(A) \) [Not Equal].

(b) The **image** (range) of a map \( f : V \to W \) is the set \( \{ f(v) \mid v \in V \} \), this can be drawn as:

![Image of f](image)

The kernel are the elements in the vector space \( V \) such that \( f(v) = 0 \), that is all the elements in \( V \) of \( f : V \to W \) which get mapped to the zero vector.

The **dimension** of the image (range) of \( f \) is called the **rank** of \( f \).

The **dimension** of the kernel of \( f \) is called the **nullity** of \( f \).

Let \( n \) be the dimension of the given vector space \( V \). The Rank-Nullity theorem states:

\[
\text{rank}(f) + \text{nullity}(f) = n
\]

(c) Since \( \text{nullity}(f) = 1 \) therefore \( f \) is **not** injective (not one-to-one) because by:

Proposition (6.17). Let \( T : V \to W \) be a linear transformation. \( T \) is one to one (injective) \( \iff \text{nullity}(T) = 0 \).

For \( f \) to be injective (one-to-one) we need \( \text{nullity}(f) = 0 \) but we have \( \text{nullity}(f) = 1 \) which means \( f \) is **not** one-to-one.

By substituting \( \text{nullity}(f) = 1 \) into \( \text{rank}(f) + \text{nullity}(f) = n \) we have

\[
\text{rank}(f) = n - 1
\]

Since \( f : P_2 \to \mathbb{R}^3 \) therefore \( n = 3 \) (dimension of \( P_2 \)) and \( \text{rank}(f) = 3 - 1 = 2 \).

Dimension of \( \mathbb{R}^3 \) is also 3. In our case we have \( \text{rank}(f) = 2 \) but \( \dim(\mathbb{R}^3) = 3 \)

This means that \( f \) is **not** surjective (not onto) because by:

Proposition (6.20). Let \( T : V \to W \) be a linear transformation. Then \( T \) is onto (surjective) \( \iff \text{rank}(T) = \dim(W) \).
(d) To prove that the given map is injective we need to show that \( \ker(f) = \{0\} \). What is \( \ker(f) \) equal to?

It is those elements \((x, y, z, t)\) in \(\mathbb{R}^4\) such that
\[
f(x, y, z, t) = (x + y, 0, z + t) = (0, 0, 0)
\]
Writing this in conventional manner we have
\[
\begin{bmatrix}
x \\
y \\
z \\
t
\end{bmatrix} = \begin{bmatrix}
x + y \\
0 \\
z + t
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
From the first and last rows we have
\[
x + y = 0 \quad \text{gives} \quad x = -y
\]
\[
z + t = 0 \quad \text{gives} \quad z = -t
\]
Let \(y = a\) and \(t = b\) where \(a\) and \(b\) are any real numbers then \(x = -y = -a\) and \(z = -t = -b\). Hence elements in \(\mathbb{R}^4\) which are mapped to the zero vectors are \((-a, a, -b, b)\) where \(a\) and \(b\) are any real numbers. Thus
\[
\ker(f) = \begin{bmatrix}
-a \\
a \\
-b \\
b
\end{bmatrix} = a\begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix} + b\begin{bmatrix}
0 \\
0 \\
-1 \\
1
\end{bmatrix} \neq 0
\]
(does not equal zero) therefore \(f\) is not injective.

A basis for \(\ker(f)\) is \(\begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
-1 \\
1
\end{bmatrix}\). Since we have two linearly independent vectors therefore the dimension of the kernel is 2 which means \(\text{nullity}(f) = 2\).

What is the dimension of \(\mathbb{R}^4\)?

4. Using the dimension theorem with \(n = 4\) and \(\text{nullity}(f) = 2\) we have
\[
\text{Rank}(f) + \text{Nullity}(f) = n
\]
\[
\text{Rank}(f) + 2 = 4 \Rightarrow \text{Rank}(f) = 2
\]
Since we are given that \(f : \mathbb{R}^4 \rightarrow \mathbb{R}^3\) and \(\dim(\mathbb{R}^3) = 3\) but we have \(\text{Rank}(f) = 2\) therefore \(f\) is not onto (not surjective).

This means that the dimension of image (range) is 2. A basis for the image can be evaluated by:
\[
f\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad f\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]
A basis for the image is \[
\begin{bmatrix}
1 \\ 0 \\ 0
\end{bmatrix}, \begin{bmatrix}
0 \\ 1 \\ 0
\end{bmatrix}.
\]

20. Do you remember what \( T_1 \circ T_2 \) means?

Let \( \mathbf{u} \) be a vector in the domain, \( \mathbb{R}^2 \), of \( T_2 \) then \( (T_1 \circ T_2)(\mathbf{u}) = T_1(T_2(\mathbf{u})) \).

Let \( \mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \) then
\[
(T_1 \circ T_2)(\mathbf{u}) = T_1 \left( T_2 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \right)
\]
\[
= T_1 \left( \begin{bmatrix} x \\ x + 2y \end{bmatrix} \right)
\]
Because \( T_2 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ x + 2y \end{bmatrix} \)
\[
= \begin{bmatrix} 2x + x + 2y \\ -x + x + 2y \end{bmatrix}
\]
Because \( T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x + y \\ -x + y \end{bmatrix} \)
\[
= \begin{bmatrix} 3x + 2y \\ 2y \end{bmatrix}
\]

What is the standard matrix \( S \) for this transformation?
\[
S = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}
\]

Selection C is correct.

21. The standard matrix \( \mathbf{B} \) for the given linear transformation
\[
S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + 5x_2 - x_3 \\ 4x_2 + 3x_3 \\ x_1 - x_2 + 4x_3 \end{bmatrix}
\]
is given by reading off the coefficients of \( x_1, \ x_2 \) and \( x_3 \):
\[
\mathbf{B} = \begin{pmatrix}
3 & 5 & -1 \\
0 & 4 & 3 \\
1 & -1 & 4
\end{pmatrix}
\]
Because \( S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + 5x_2 - x_3 \\ 4x_2 + 3x_3 \\ x_1 - x_2 + 4x_3 \end{bmatrix} \)

Therefore the matrix \( \mathbf{C} \) is given by
\[
\mathbf{C} = (S \circ T)(\mathbf{x})
= S(T(\mathbf{x}))
= \mathbf{B}(\mathbf{Ax})
= (\mathbf{BA})\mathbf{x} = \begin{pmatrix}
3 & 5 & -1 \\
0 & 4 & 3 \\
1 & -1 & 4
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
9 & -6 & 15 \\
20 & -1 & 0 \\
15 & 5 & 5
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
Hence matrix $C = BA = \begin{pmatrix} 9 & -6 & 15 \\ 20 & -1 & 0 \\ 15 & 5 & 5 \end{pmatrix}$.

22. (a) Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^4$. To prove that $\phi$ is a linear mapping (transformation) we need to show both the following conditions:

\[ \phi(\mathbf{x} + \mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y}) \quad \text{and} \quad \phi(k\mathbf{x}) = k\phi(\mathbf{x}) \quad (k \text{ is scalar}) \]

We have

\[ \phi(\mathbf{x} + \mathbf{y}) = \mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} = \phi(\mathbf{x}) + \phi(\mathbf{y}) \]

Let $k$ be scalar then $\phi(k\mathbf{x}) = \mathbf{A}(k\mathbf{x}) = k\mathbf{Ax} = k\phi(\mathbf{x})$.

Since we have $\phi(\mathbf{x} + \mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$ and $\phi(k\mathbf{x}) = k\phi(\mathbf{x})$ therefore $\phi$ is a linear map (transformation).

We are given that $\mathbf{e}_1 = (1, 0, 0, 0)^T$ therefore

\[ \phi(\mathbf{e}_1) = \mathbf{Ae}_1 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -1 & -5 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \]

(b) Since we need to find the determinant of a $4 \times 4$ matrix therefore it should easier to first carry out some simple row operations. Labelling the rows of matrix $\mathbf{A}$ we have

\[ \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4 \]

Carrying out the row operations $\mathbf{R}_2 - \mathbf{R}_1$, $\mathbf{R}_3 - \mathbf{R}_1$ and $\mathbf{R}_4 + \mathbf{R}_1$ we have

\[ \mathbf{R}_1^*, \mathbf{R}_2^*, \mathbf{R}_3^*, \mathbf{R}_4^* \]

Carrying out the row operations $\mathbf{R}_2 - \mathbf{R}_1$, $\mathbf{R}_3 - \mathbf{R}_1$ and $\mathbf{R}_4 + \mathbf{R}_1$ we have

\[ \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & 5 & 1 & 3 \end{pmatrix} \]

The determinant of this last matrix can be found by expanding along the first column:
Expanding along the first row

\[
\begin{vmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 4 & 3 \\
0 & 5 & 1 & 3 \\
\end{vmatrix} = 1 \begin{vmatrix}
1 & 2 & 0 \\
2 & 4 & 3 \\
5 & 1 & 3 \\
\end{vmatrix} = \det \begin{pmatrix}
4 & 3 \\
1 & 3 \\
\end{pmatrix} - 2 \det \begin{pmatrix}
2 & 3 \\
5 & 3 \\
\end{pmatrix} = (12 - 3) - 2(6 - 15) = 9 - 2(-9) = 27
\]

All the row operations carried out in (†) do not change the determinant therefore \( \det(A) = 27 \).

(c) For a basis for \( \ker(\phi) \) we need to place matrix A into a reduced row echelon matrix \( R \) and then solve the homogeneous system \( Rx = O \). From (†) in part (b) we have

\[
\begin{align*}
R_1 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 4 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_2 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 4 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_3 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 4 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_4 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 4 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix}
\end{align*}
\]

Carrying out the row operation \( R_3 = -2R_2 \) yields

\[
\begin{align*}
R_1 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_2 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_3 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix} \\
R_4 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 5 & 1 & 3 \\
\end{pmatrix}
\end{align*}
\]

Executing the row operations \( R_4 = R_1 \) and \( R_4'/3 \) gives

\[
\begin{align*}
R_1 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 5 & 1 & 0 \\
\end{pmatrix} \\
R_2 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 5 & 1 & 0 \\
\end{pmatrix} \\
R_3 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 5 & 1 & 0 \\
\end{pmatrix} \\
R_4 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 5 & 1 & 0 \\
\end{pmatrix}
\end{align*}
\]

Carrying out the row operation \( R_4 = -5R_2 \) gives

\[
\begin{align*}
R_1 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9 & 0 \\
\end{pmatrix} \\
R_2 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9 & 0 \\
\end{pmatrix} \\
R_3 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9 & 0 \\
\end{pmatrix} \\
R_4 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9 & 0 \\
\end{pmatrix}
\end{align*}
\]

Dividing the bottom row by \(-9\) yields

\[
\begin{align*}
R_1 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \\
R_2 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \\
R_3 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \\
R_4 & \begin{pmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\end{align*}
\]

Carrying out the row operations \( R_1 = R_3 \), \( R_1 + R_4 \) and \( R_2 = -2R_4 \) gives
Interchanging the bottom two rows gives the reduced row echelon form matrix $R$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= R
$$

Solving $Rx = O$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

gives $x_1 = x_2 = x_3 = x_4 = 0$

This means that $\ker(\phi) = O$ and so there is no basis for $\ker(\phi)$.

Since $\ker(\phi) = O$ therefore $nullity(\phi) = 0$ and we are given that $\phi : \mathbb{F}^4 \rightarrow \mathbb{F}^4$

therefore $im(\phi) = \mathbb{F}^4$ and a basis for $\mathbb{F}^4$ is the standard basis:

$$
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

The given linear map $\phi$ is invertible because matrix $A$ which represents $\phi$ is invertible since $\det(A) = 27 \neq 0$ [Not equal to zero].

23. (a) (i) Let $u$ and $v$ be vectors in $V$ then $T$ is a linear transformation if $T(u + v) = T(u) + T(v)$ and $T(au) = aT(u)$ where $k$ is a scalar.

(ii) The matrix $A$ representing $T$ with respect to the basis $B = \{u_1, \ldots, u_n\}$ is

$$
A = \begin{bmatrix}
a_{i1} & \cdots & a_{in} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix}
$$

where $T(u_i) = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\ldots$, $T(u_n) = \begin{bmatrix} a_{in} \\ \vdots \\ a_{mn} \end{bmatrix}$.

(b) We are given that $B = \{\sin x, \cos x\}$ which are the basis vectors. We need to find $T(\sin x)$, $T(\cos x)$ and write these in terms of $\begin{bmatrix} a \sin x \\ b \cos x \end{bmatrix}$.

We have
\( T(\sin x) = (\sin x)'+(\sin x)^* \)  
\[ \text{[Because} \quad T(f(x)) = f'(x) + f^*(x) \text{]} \]
\[ = \cos x + (\cos x)' \]
\[ = \cos x - \sin x \]
Similarly we have
\[ T(\cos x) = (\cos x)'+(\cos x)^* \]  
\[ \text{[Because} \quad T(f(x)) = f'(x) + f^*(x) \text{]} \]
\[ = -\sin x + (-\sin x)' \]
\[ = -\sin x - \cos x \]
Collecting the above we have
\[ T(\sin x) = \cos x - \sin x = (-1)\sin x + (1)\cos x \quad \Rightarrow \quad [T(\sin x)]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]
\[ T(\cos x) = -\sin x - \cos x = (-1)\sin x + (-1)\cos x \quad \Rightarrow \quad [T(\cos x)]_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \]

**What is the matrix representation of \( T \)?**

Let \([T]_B\) be the matrix representing the given linear transformation \( T \) with respect to the basis \( B = \{ \sin x, \cos x \} \):
\[ [T]_B = \begin{bmatrix} [T(\sin x)]_B & [T(\cos x)]_B \end{bmatrix} \]
\[ = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \]

(c) By Proposition (6.32) we have \( T \) is invertible \( \iff \) the matrix \([T]_B\) is invertible.

**How do we determine whether \([T]_B = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \) is invertible?**

Check that the determinant is **not** equal to zero:
\[ \det \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = 1 + 1 = 2 \neq 0 \]

Thus the given linear transformation is invertible and taking the inverse gives
\[ [T]^{-1}_B = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \]

(d) To find \( f(x) \) such that \( f''(x) + f'(x) = 2\sin x + 3\cos x \) we need to use \([T]^{-1}_B\)

found in part (c) and the vector \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) because we are given \( 2\sin x + 3\cos x \):
\[ [T]^{-1}_B \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -5 \end{bmatrix} \]

Hence \( f(x) = \frac{1}{2} \sin x - \frac{5}{2} \cos x \).
24. How do we show that \( \langle u, v \rangle := \langle T(u), T(v) \rangle \) is an inner product?

We need to use the following from Chapter 5:

**Definition (5.1).**

An inner product on a real vector space \( V \) is an operation which assigns to each pair of vectors, \( u \) and \( v \), a unique real number \( \langle u, v \rangle \) which satisfies the following axioms for all vectors \( u, v \) and \( w \) in \( V \) and all scalars \( k \).

(i) \( \langle u, v \rangle = \langle v, u \rangle \) [Commutative Law]

(ii) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \) [Distributive Law]

(iii) \( \langle ku, v \rangle = k \langle u, v \rangle \)

(iv) \( \langle u, u \rangle \geq 0 \) and we have \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \)

**Proof.**

Let \( u, v, w \in V \) and we are given that \( T : V \to \mathbb{R}^n \). The standard inner product on \( \mathbb{R}^n \) is the dot or scalar product denoted by \( \cdot \). We have

\[
\langle u, v \rangle = \langle T(u), T(v) \rangle = T(u) \cdot T(v)
\]

Checking (i):
We have

\[
\langle u, v \rangle = \langle T(u), T(v) \rangle = T(u) \cdot T(v)
\]

Because \( \cdot \) is an Inner Product

Thus part (i) of Definition (5.1) is satisfied.

Checking (ii):
We have

\[
\langle u + v, w \rangle = \langle T(u + v), T(w) \rangle = T(u + v) \cdot T(w)
\]

Because \( T \) is linear

\[
= [T(u) + T(v)] \cdot T(w) = [T(u) \cdot T(w)] + [T(v) \cdot T(w)]
\]

[Because \( \cdot \) is an I.P.]

\[
= \langle u, v \rangle + \langle v, w \rangle
\]

Hence part (ii) is satisfied.

Checking (iii):
Let \( k \) be a scalar. We have

\[
\langle ku, v \rangle = \langle T(ku), T(v) \rangle
\]

[Because \( T \) is linear]

\[
= kT(u) \cdot T(v) = k [T(u) \cdot T(v)] = k \langle u, v \rangle
\]

Part (iii) is satisfied.

Checking (iv):
We have

\[
\langle u, u \rangle = \langle T(u), T(u) \rangle = T(u) \cdot T(u) \geq 0
\]

[Because \( \cdot \) is an inner product]

Also
0 = \langle u, u \rangle = \langle T(u), T(u) \rangle = T(u) \cdot T(u) \Rightarrow T(u) = O

Since we are given that $T$ is one-to-one therefore $\ker(T) = O$ because Proposition (6.15) says:

$T$ is one-to-one $\iff \ker(T) = O$

This means that $T(u) = O$ implies $u = O$. Part (iv) is satisfied.

Since all four parts of definition (5.1) is satisfied therefore $\langle u, v \rangle_v := \langle T(u), T(v) \rangle$ is an inner product.

25. We need to prove that if $v_1$ and $v_2$ are linear independent vectors in $V$ where $T: V \rightarrow W$ is a linearly one-to-one transformation then $T(v_1)$ and $T(v_2)$ are linearly independent.

**Proof.**

Consider the linear combination

$c_1 T(v_1) + c_2 T(v_2) = O$

where $c_1$ and $c_2$ are scalars. Since $T$ is linear we have

$T(c_1 v_1 + c_2 v_2) = O$

We are also given that $T$ is one-to-one which means that $\ker(T) = O$ because Proposition (6.15) says that:

$T: V \rightarrow W$ is one-to-one $\iff \ker(T) = O$

Therefore from $T(c_1 v_1 + c_2 v_2) = O$ we have

$c_1 v_1 + c_2 v_2 = O$

We are given that $v_1$ and $v_2$ are linearly independent which implies that $c_1 = c_2 = 0$. Thus we have $c_1 T(v_1) + c_2 T(v_2) = O \Rightarrow c_1 = c_2 = 0 \Rightarrow T(v_1)$ and $T(v_2)$ are linearly independent. This is our required result.

26. The following is true and proof of this follows:

If $V$ is a vector space and $T: V \rightarrow V$ is an injective linear transformation, then $T$ is surjective.

**Proof.**

Since $T$ is injective which is another term for one-to-one therefore $\ker(T) = O$ because Proposition (6.15) says:

$T$ is one-to-one $\iff \ker(T) = O$

This means that the dimension of $\ker(T)$ is zero. Using the Dimension Theorem (6.12) which states

$\dim(\ker(T)) + \dim(\text{range}(T)) = n$ (n is the dimension of $V$)

We have

$0 + \dim(\text{range}(T)) = \dim(V)$

Thus $\dim(\text{range}(T)) = \dim(V)$. By
Proposition (6.20). Let \( T : V \to W \) be a linear transformation. Then \( T \) is onto \( \iff \text{rank} \, (T) = \dim(W) \).

We conclude \( T \) is onto or surjective because \( \dim(\text{range}(T)) = \text{rank} \, (T) = \dim(V) \).

27. (a) We need to show that \( \text{Im}(A) \subseteq \ker(A) \) given \( A^2 = 0 \) and \( A : V \to V \).

\textbf{Proof.}

Suppose \( \text{Im}(A) \not\subseteq \ker(A) \) that is \( \text{Im}(A) \) is \textbf{not} subset of \( \ker(A) \). This means that there is a vector \( y \in \text{Im}(A) \) such that \( y \not\in \ker(A) \). Since \( y \in \text{Im}(A) \) therefore there is a vector \( x \in V \) such that \( A(x) = y \). Consider \( A^2(x) \):

\[
A^2(x) = (A \circ A)(x) = A(A(x)) = A(y) \quad \text{[Because} \ A(x) = y\text{]} \neq 0 \quad \text{[Because} \ y \not\in \ker(A)\text{]}
\]

Remember the definition of \( \ker(A) \) are those elements \( u \) in \( V \) such that \( A(u) = 0 \).

Since \( y \not\in \ker(A) \) therefore \( A(y) \neq 0 \).

We have \( A^2(x) \neq 0 \) which means that \( A^2 \neq 0 \). This is a contradiction because we are given that \( A^2 = 0 \) therefore our supposition \( \text{Im}(A) \not\subseteq \ker(A) \) must be wrong so \( \text{Im}(A) \subseteq \ker(A) \) which is our required result.

\( \blacksquare \)

(b) We need to show that the rank of \( A \) is at most 5.

The \( \text{rank} \, (A) = \dim(\text{Im}(A)) \) and \( \text{nullity}(A) = \dim(\ker(A)) \).

\textbf{Proof.}

From part (a) we have \( \text{Im}(A) \subseteq \ker(A) \) which means that

\[
\dim(\text{Im}(A)) \leq \dim(\ker(A)) \quad \text{or} \quad \text{rank} \, (A) \leq \text{nullity}(A)
\]

By the rank-nullity theorem which is (6.12) we have

\[
\text{rank} \, (A) + \text{nullity}(A) = \dim(V) = 10
\]

Suppose \( \text{rank} \, (A) > 5 \) then

\[
\text{nullity}(A) = 10 - \text{rank} \, (A) \leq 4 \quad \text{because we are supposing} \ \text{rank} \, (A) > 5
\]

This means that \( \text{rank} \, (A) > \text{nullity}(A) \). This is impossible because in the above we had \( \text{rank} \, (A) \leq \text{nullity}(A) \). Thus our supposition \( \text{rank} \, (A) > 5 \) must be wrong so \( \text{rank} \, (A) \leq 5 \) which is our required result.

\( \blacksquare \)
28. Let \(A, B\) and \(C\) be in \(M_{nn}\). Then \(T\) is linear if \(T(A + C) = T(A) + T(C)\) and \(T(kA) = kT(A)\) where \(k\) is a scalar. Checking the first result:

\[
T(A + C) = (A + C)B + B(A + C) = AB + CB + BA + BC
\]

We have \(T(A + C) = T(A) + T(C)\).

Checking \(T(kA) = kT(A)\):

\[
T(kA) = (kA)B + B(kA) = k(AB) + k(BA) = k(AB + BA) = kT(A)
\]

Since we have \(T(A + C) = T(A) + T(C)\) and \(T(kA) = kT(A)\) therefore \(T\) is a linear transformation.

29. Let \(u = \begin{pmatrix} a \\ b \end{pmatrix}\) and \(v = \begin{pmatrix} c \\ d \end{pmatrix}\). To show that \(T\) is not linear we prove

\[
T(u + v) \neq T(u) + T(v)
\]

We have

\[
T(u + v) = T\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = T\left(\begin{pmatrix} a + c \\ b + d \end{pmatrix}\right) = \begin{pmatrix} e^{a+c} \\ e^{b+d} \end{pmatrix}
\]

Because

\[
T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} e^x \\ e^y \end{pmatrix}
\]

[Using the rules of indices]

\[
T(u) + T(v) = T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) + T\left(\begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} e^a \\ e^b \end{pmatrix} + \begin{pmatrix} e^c \\ e^d \end{pmatrix} = \begin{pmatrix} e^a + e^c \\ e^b + e^d \end{pmatrix} \neq \begin{pmatrix} e^{a+c} \\ e^{b+d} \end{pmatrix}
\]

Thus \(T(u + v) \neq T(u) + T(v)\) which means that \(T\) is not a linear transformation.