Complete Solutions to Exercises 7.5

1. (a) The eigenvalues and eigenvectors of matrix $A^TA$ are

$$\lambda_1 = 1, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 4, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The singular values of matrix $A$ are $\sigma_1 = \sqrt{1} = 1$ and $\sigma_2 = \sqrt{4} = 2$. Using $\sigma_i u_i = A v_i$ and $\sigma_2 u_2 = A v_2$ we have

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Hence from this last result $2u_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ we have $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We have $U = (u_1 \quad u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $V = (v_1 \quad v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Note that $U = V = I$. The triple factorization of the given matrix $A$ is

$$A = U D V^T = I D = D$$

(b) The eigenvalues and normalized eigenvectors of matrix $A^TA$ are

$$\lambda_1 = 81, \quad v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 1, \quad v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The singular values of matrix $A$ are $\sigma_1 = \sqrt{81} = 9$ and $\sigma_2 = 1$. Using $\sigma_i u_i = A v_i$ and $\sigma_2 u_2 = A v_2$ we have

$$9u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

We have $u_1 = \frac{1}{9\sqrt{5}} \begin{pmatrix} 9 \\ 18 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Substituting this and $u_2$ gives

$$U = (u_1 \quad u_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$V^T = (v_1 \quad v_2)^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$ 

You may like to check the factorization $A = U D V^T$.

(c) The eigenvalues and normalized eigenvectors of matrix $A^TA$ are

$$\lambda_1 = 6, \quad v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 1, \quad v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The singular values of matrix $A$ are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = 1$.

Using $u_1 = \frac{1}{\sigma_1} A v_1$ and $u_2 = \frac{1}{\sigma_2} A v_2$ we have
\[ u_1 = \frac{1}{\sqrt{6}} Av_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } u_2 = Av_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ \sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \]

Since A is a 3 by 2 matrix so U is a 3 by 3 matrix which means we need to find the vector \( u_3 \) which is orthogonal to both \( u_1 \) and \( u_2 \):

\[
\begin{pmatrix} 1 & 2 & 5 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 1, \ y = 2, \ z = -1 \Rightarrow u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}
\]

Normalizing the vector \( u_3 \) gives

\[ u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \]

We have \( u_1 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \), \( u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \) and \( u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \).

Substituting these and the above into \( U, D \) and \( V \) gives:

\[ U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \] \quad \text{and} \quad \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 0 \end{pmatrix}
\]

\[ V^T = (v_1, v_2)^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \]

You may like to check the factorization \( A = UDV^T \).

(d) The product \( A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{pmatrix} \). The eigenvalues and normalized eigenvectors of matrix \( A^T A \) are

\[ \lambda_1 = 6, \quad v_1 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \quad \lambda_2 = 1, \quad v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_3 = 0, \quad v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \]

The singular values of matrix A are \( \sigma_1 = \sqrt{6} \) and \( \sigma_2 = 1 \).

Using \( u_1 = \frac{1}{\sigma_1} Av_1 \) and \( u_2 = \frac{1}{\sigma_2} Av_2 \) we have
\[ u_1 = \frac{1}{\sqrt{6}} Av_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{180}} \begin{pmatrix} 6 \\ 12 \end{pmatrix} \]

\[ u_2 = \frac{1}{1} Av_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \]

Since \( A \) is a 2 by 3 matrix so \( U \) is a 2 by 2 matrix, \( D \) is a 2 by 3 matrix and \( V \) is a 3 by 3 matrix:

Substituting these and the above into \( U, D \) and \( V \) gives:

\[ U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 6/\sqrt{180} & -2/\sqrt{5} \\ 12/\sqrt{180} & 1/\sqrt{5} \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{50} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \]

You may like to check the factorization \( A = UDV^T \) by first transposing matrix \( V \).

(e) The product \( A^T A = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix} \). The eigenvalues and normalized eigenvectors of matrix \( A^T A \) are

\[ \lambda_1 = 20, \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 0, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

The positive singular value of matrix \( A \) is \( \sigma_1 = \sqrt{20} \).

Using \( u_1 = \frac{1}{\sigma_1} Av_1 \) we have

\[ u_1 = \frac{1}{\sqrt{20}} Av_1 = \frac{1}{\sqrt{20}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{40}} \begin{pmatrix} 2 \\ \sqrt{6} \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{[Because} \sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}] \]

Since \( A \) is a 2 by 2 matrix so \( U \) is a 2 by 2 matrix. What is \( u_2 \) equal to?

\( u_1 \) needs to be orthogonal to \( u_1 \) which means \( u_1 \cdot u_1 = 0 \) therefore by inspection and normalizing we have

\[ u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \]

Substituting these and the above into \( U, D \) and \( V \) gives:

\[ U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{20} & 0 \\ 0 & 0 \end{pmatrix} \]

\[ V^T = (v_1 \ v_2)^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
You may like to check the factorization $A = UDV^T$.

(f) The product $A^T A = \begin{pmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix}$. The eigenvalues and normalized eigenvectors of matrix $A^T A$ are

$$\lambda_1 = 30, \quad \mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 0, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_3 = 0, \quad \mathbf{v}_3 = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

The positive singular value of matrix $A$ is $\sigma_1 = \sqrt{30}$. Using $u_i = \frac{1}{\sigma_i} \mathbf{Av}_i$ we have

$$u_1 = \frac{1}{\sqrt{20}} A \mathbf{v}_1 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{90}} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{[Because } \sqrt{90} = \sqrt{9 \times 10} = 3\sqrt{10}]$$

Since $A$ is a 2 by 3 matrix so $U$ is a 2 by 2 matrix, $D$ is a 2 by 3 matrix and $V$ is a 3 by 3 matrix. What is $u_3$ equal to?

$u_2$ needs to be orthogonal to $u_1$. We need $u_2 \cdot u_1 = 0$

As in part (e) we have $u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. We have

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} \sqrt{30} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V^T = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}^T$$

You may like to check the factorization $A = UDV^T$.

2. We need to prove that:

Let $A$ be any matrix. Then the eigenvalues of $A^T A$ are positive or zero.

**Proof.**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A^T A$ with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ respectively. For an arbitrary eigenvector $\mathbf{v}_j$ we have

$$A^T A \mathbf{v}_j = \lambda_j \mathbf{v}_j \quad (*)$$

By Proposition (7.31):

(7.31). Let $A$ be any matrix. Then $A^T A$ is a symmetric matrix.
This means that $A^TA$ can be orthogonally diagonalized so the eigenvectors are orthonormal.

Consider the norm square $\|Av\|^2$:

$$\|Av\|^2 = Av \cdot Av = (Av)^TAv$$

$$= v_j^TA^TAv_j$$

(by (*)

$$= v_j^T\lambda_jv_j = \lambda_j(v_j \cdot v_j) = \lambda_j$$  [Because $(v_j \cdot v_j) = \|v_j\| = 1$]

We have $\lambda_j = \|Av_j\|^2 \geq 0$. Hence all the eigenvalues of $A^TA$ are positive or zero.

3. Required to prove:

Let matrix $A$ have $k$ positive singular values. Then the rank of matrix $A$ is $k$.

Proof.

Let $A$ be a $m \times n$ matrix. By SVD:

(7.30). We can decompose any given matrix $A$ of size $m \times n$ with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$ where $k \leq m$, into $UDV^T$, that is

$$A = UDV^T$$

where $U$ is a $m \times m$ orthogonal matrix, $D$ is a $m \times n$ matrix and $V$ is an $n \times n$ orthogonal matrix.

We have $A = UDV^T$ where $U$ and $V$ are orthogonal matrices. Since $U$ is orthogonal so it is invertible because $U^{-1} = U^T$ and similarly $V$ is orthogonal so $V^{-1} = V^T$. Hence $(V^T)^{-1} = (V^{-1})^{-1} = V$ which means that $V^T$ is invertible. By hint we have:

$$\text{rank}(A) = \text{rank}(UDV^T)$$

$$= \text{rank}(DV^T) = \text{rank}(D) = k$$

This completes our proof.

4. (a) We need to prove that $\{u_1, u_2, \ldots, u_k\}$ form an orthonormal basis for the column space of matrix $A$.

Proof.

$U$ is a $m \times m$ orthogonal matrix so the column vectors of matrix $U$ are orthonormal:

$$U = (u_1 \ u_2 \ \ldots \ u_n)$$

By result of question 3 we have the rank of matrix $A$ is $k$.

By Proposition (3.29) of chapter 3:

(3.29). $\text{rank}(A) = \text{Row rank of } A = \text{Column rank of } A$

Hence the dimension of the column space is $k$ which means we need $k$ basis vectors for the column space of matrix $A$.

We are given that $\sigma_1, \sigma_2, \ldots, \sigma_k$ are positive and from the main theorem of the section (7.30) for $j = 1, 2, 3, \ldots, k$ we have

$$u_j = \frac{1}{\sigma_j}Av_j$$
By Proposition (3.36):

(3.36). The linear system $Ax = b$ has a solution $\iff b$ can be generated by the column space of matrix $A$.

Therefore $u_j$ is in the column space of matrix $A$. We have $k$ orthonormal vectors

\[ \{u_1, u_2, \ldots, u_k\} \]

which are in the column space of matrix $A$. Since orthogonal vectors are linearly independent so they $\{u_1, u_2, \ldots, u_k\}$ form an orthonormal basis for the column space of matrix $A$.

(b) We need to prove that $\{v_1, v_2, \ldots, v_k\}$ form an orthonormal basis for the row space of matrix $A$.

Proof.

Since $V$ is an $n$ by $n$ orthogonal matrix whose columns are the eigenvectors of $A^T A$:

\[ V = (v_1 \ v_2 \ \cdots \ v_n) \]

These eigenvectors are an orthonormal set of vectors because $V$ is orthogonal.

By result of question 3 we have the rank of matrix $A$ is $k$. By Definition (3.28) of chapter 3:

(3.28). The rank of a matrix $A$ is the row rank of $A$.

Hence the dimension of the row space is $k$ which means we need $k$ basis vectors for the row space. A subset of $k$ vectors in the above, $S = \{v_1, v_2, \ldots, v_k\}$, is also orthonormal. As these vectors are orthogonal so they are linearly independent which means they form a basis. Therefore $S$ forms an orthonormal basis for the row space of matrix $A$.

(c) Required to prove:

The set of vectors $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ form an orthonormal basis for the null space of matrix $A$.

Proof.

By Theorem (3.34):

(3.34). If $A$ is a matrix with $n$ columns (number of unknowns) then

\[ \text{nullity}(A) + \text{rank}(A) = n \]

We have the dimension of null space is $n - k$ and there are $n - k$ vectors in the set $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Remember this set $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ represent the orthonormal eigenvectors of $A^T A$ which correspond to the zero eigenvalues $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_k = 0$.

This means that for $j = k + 1, k + 2, \ldots, k + n$ we have

\[ (A^T A) v_j = \lambda_j v_j = 0 v_j = 0 \]

These vectors $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ form an orthonormal basis for the null space of $A^T A$.

The null space of $A^T A$ and $A$ are identical. Hence $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ form an orthonormal basis for the null space of matrix $A$.

5. Proof.

The eigenvalues of $A^T A$ are unique. Why?

By Question 9 of Exercises 7.2:
Let $A$ be a square matrix and $\lambda$ be an eigenvalue with the corresponding eigenvector $u$. The eigenvalue $\lambda$ is unique for the eigenvector $u$.

The singular values are given by the positive roots:

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \ldots, \quad \sigma_n = \sqrt{\lambda_n}$$

Therefore the singular values are unique.

6. We need to prove that the column vectors of matrix $U$ in $A = UDV^T$ are the eigenvectors of $AA^T$.

Proof.

Using the singular value decomposition $A = UDV^T$ we have

$$AA^T = (UDV^T)(UDV^T)^T$$

$$= (UDV^T)(V^TD^TU^T)$$

[By using $(XYZ)^T = Z^TY^TX^T$]

$$= UDV^TV^TD^TU^T$$

[Because $(X^T)^T = X$]

$$= U(DD^T)U^T = U(D^T)U^T$$

where $D^T = DD^T$ is a diagonal matrix.

Remember $U$ is an orthogonal matrix so it inverse is given by $U^T$. Left-multiplying the above result $AA^T = U(D^T)U^T$ by $U^T$ and right-multiplying by $U$ gives

$$U^T(AA^T)U = U^TU(D^T)U^TU = D$$

Since $U^T(AA^T)U = D^T$, the matrix $U$ diagonalizes $AA^T$ and the columns of $U$ are the eigenvectors of $AA^T$. This completes our proof.

7. Required to prove that:

The singular values of $A$ and $A^T$ are identical.

Proof.

The singular values of a matrix $A$ are given by the square roots of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $AA^T$

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \ldots, \quad \sigma_n = \sqrt{\lambda_n}$$

The singular values of a matrix $A^T$ are given by the square roots of the eigenvalues $t_1, t_2, \ldots, t_n$ of $(AA^T)^T$.

By Question 16 of Exercise 7.2:

The eigenvalues of the transposed matrix, $A^T$, are exactly the eigenvalues of the matrix $A$.

Hence $(AA^T)^T$ will have the same eigenvalues as $AA^T$ which means:

$$t_1 = \lambda_1, \quad t_2 = \lambda_2, \ldots, \quad t_n = \lambda_n$$

Therefore the singular values of $A^T$ are the same. Hence the singular values of both $A$ and the transposed matrix $A^T$ are identical.

8. (a) We have to prove that:
The set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \} \) form an orthonormal basis for the range of \( T \).

**Proof.**

Let \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \} \) be the first \( k \) column vectors of matrix \( \mathbf{U} \). By Proposition (7.34) part(a):

(7.34) (a) The set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \} \) is an orthonormal basis for the column space of matrix \( \mathbf{A} \).

Also we are given that \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) so by Proposition (5.13) of chapter 5:

(5.13). Let \( T: \mathbb{R}^m \to \mathbb{R}^n \) be a linear transformation given by \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \). Then \( \text{range}(T) \) is the column space of \( \mathbf{A} \).

So the range of the transformation \( T \) is the column space of matrix \( \mathbf{A} \) therefore \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \} \) is an orthonormal basis for the range.

\[ \blacksquare \]

(b) Required to prove that:

The set of vectors \( \{ \mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n \} \) form an orthonormal basis for the kernel of \( T \).

**Proof.**

Remember the kernel of a transformation \( T \) are the vectors in the start vector space which are mapped to the zero vector. In our case we have \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) so it is the vectors \( \mathbf{x} \) which satisfy \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \mathbf{0} \). Of course this is the null space of matrix \( \mathbf{A} \). By Proposition (7.34):

(7.34) (c) The set of vectors \( \{ \mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n \} \) form an orthonormal basis for the null space of matrix \( \mathbf{A} \).

Hence \( \{ \mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_n \} \) is an orthonormal basis for kernel of \( T \).

\[ \blacksquare \]