Complete Solutions to Exercises 6.2

1. We use the formula for the determinant of a 3×3 matrix:
   (a) Expand along the middle row because it contains a zero:
   \[
   \det \begin{pmatrix}
   1 & 3 & -1 \\
   2 & 0 & 5 \\
   -6 & 3 & 1
   \end{pmatrix}
   = -2\det \begin{pmatrix}
   3 & -1 \\
   3 & 1
   \end{pmatrix} + 0\det \begin{pmatrix}
   1 & -1 \\
   -6 & 1
   \end{pmatrix} - 5\det \begin{pmatrix}
   1 & 3 \\
   -6 & 3
   \end{pmatrix}
   \]
   \[
   = -2[(3\times1)-(3\times(-1))] + 0 - 5[(1\times3)-((-6)\times3)] = -117
   \]
   by (6.1)
   (b) Similarly we have
   \[
   \det \begin{pmatrix}
   2 & -10 & 11 \\
   5 & 3 & -4 \\
   7 & 9 & 12
   \end{pmatrix}
   = 2\det \begin{pmatrix}
   3 & -4 \\
   9 & 12
   \end{pmatrix} + 10\det \begin{pmatrix}
   5 & -4 \\
   7 & 12
   \end{pmatrix} + 11\det \begin{pmatrix}
   5 & 3 \\
   7 & 9
   \end{pmatrix}
   \]
   \[
   = 2[(3\times12)-(9\times(-4))] + 10[(5\times12)-(7\times(-4))] + 11[(5\times9)-(7\times3)] = 1288
   \]
   by (6.1)
   (c) Very similar to parts (a) and (b). Thus \( \det (C) = -114 \).

2. Using (6.6):
   \[
   \det \begin{pmatrix}
   i & j & k \\
   7 & 3 & -2 \\
   4 & 2 & 7
   \end{pmatrix}
   = i\det \begin{pmatrix}
   3 & -2 \\
   4 & 7
   \end{pmatrix} - j\det \begin{pmatrix}
   7 & -2 \\
   4 & 7
   \end{pmatrix} + k\det \begin{pmatrix}
   7 & 3 \\
   4 & 2
   \end{pmatrix}
   \]
   \[
   = i[(3\times7)-(2\times(-2))] - j[(7\times7)-(4\times(-2))] + k[(7\times2)-(4\times3)]
   \]
   \[
   = 25i - 57j + 2k
   \]

3. Expand the 3 by 3 matrix as normal:
   \[
   \det \begin{pmatrix}
   1 & 0 & -3 \\
   5 & x & -7 \\
   3 & 9 & x-1
   \end{pmatrix}
   = 1\det \begin{pmatrix}
   x & -7 \\
   9 & x-1
   \end{pmatrix} - 0 - 3\det \begin{pmatrix}
   5 & x \\
   3 & 9
   \end{pmatrix}
   \]
   \[
   = \begin{vmatrix}
   x(x-1)-(9(-7)) \\
   5\times9-(3\times3)
   \end{vmatrix}
   \]
   \[
   = x^2 - x + 63 - 3[45 - 3x]
   \]
   \[
   = x^2 - x + 63 - 135 + 9x = x^2 + 8x - 72
   \]
   Since we want to find the values of \( x \) when the determinant is zero, we have to solve
   \[
   x^2 + 8x - 72 = 0
   \]

   **How do we solve this quadratic equation?**
   Use the quadratic formula with \( a = 1, \ b = 8 \) and \( c = -72 \)
   \[
   x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - (4\times1\times(-72))}}{2}
   \]
   \[
   = \frac{-8 \pm \sqrt{352}}{2} = \frac{-8 \pm 18.76}{2} = -13.38, 5.38
   \]
   Thus \( x = -13.38, 5.38 \).
4. We need to find the cofactor of each element of the matrix \( A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 3 & 7 \\ 6 & -1 & 0 \end{pmatrix} \).

Cofactor of 1 is
\[
\det \begin{pmatrix} 3 & 7 \\ -1 & 0 \end{pmatrix} = (3\times0) - (-1\times7) = 7
\]

Cofactor of 0 is
\[
-\det \begin{pmatrix} -2 & 7 \\ 6 & 0 \end{pmatrix} = -[(-2\times0) - (6\times7)] = 42
\]

Cofactor of 5 is
\[
\det \begin{pmatrix} -2 & 3 \\ 6 & -1 \end{pmatrix} = [(-2\times(-1)) - (6\times3)] = -16
\]

Cofactor of \(-2\) is
\[
-\det \begin{pmatrix} 0 & 5 \\ -1 & 0 \end{pmatrix} = -[(0\times0) - (-1\times5)] = -5
\]

Cofactor of 3 is
\[
\det \begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix} = [(1\times0) - (6\times5)] = -30
\]

Cofactor of 7 is
\[
-\det \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} = -[(1\times(-1)) - (6\times0)] = 1
\]

Cofactor of 6 is
\[
\det \begin{pmatrix} 0 & 5 \\ 3 & 7 \end{pmatrix} = [(0\times7) - (3\times5)] = -15
\]

Cofactor of \(-1\) is
\[
-\det \begin{pmatrix} 1 & 5 \\ -2 & 7 \end{pmatrix} = -[(1\times7) - (-2\times5)] = -17
\]

Cofactor of 0 is
\[
\det \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = [(1\times3) - (-2\times0)] = 3
\]

Collecting the cofactors gives the cofactor matrix:
\[
C = \begin{pmatrix} 7 & 42 & -16 \\ -5 & -30 & 1 \\ -15 & -17 & 3 \end{pmatrix}
\]

Transposing this matrix (interchanging rows and columns) gives \( C^T = \begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix} \).

The inverse matrix \( A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \) where \( \text{adj}(A) = C^T = \begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix} \).
What is the determinant of \( A \)?

\[
\det(A) = \det \begin{pmatrix} 1 & 0 & 5 \\ -2 & 3 & 7 \\ 6 & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 3 & 7 \\ -1 & 0 \end{pmatrix} + 5 \det \begin{pmatrix} -2 & 3 \\ 6 & -1 \end{pmatrix}
\]

\[
= (0 + 7) + 5(2 - 18) = -73
\]

Substituting \( \det(A) = -73 \) and \( \text{adj}(A) = \begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix} \) into \( A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \):

\[
A^{-1} = \frac{1}{73} \begin{pmatrix} 7 & -5 & -15 \\ 42 & -30 & -17 \\ -16 & 1 & 3 \end{pmatrix}
\]

5. (a) In this case \( \det(A) = 1 \) so we have an invertible matrix and use (6.2) to find the inverse.

Exchanging numbers 3 and 9 and placing a negative sign in front of the other numbers gives:

\[
A^{-1} = \begin{pmatrix} 3 & -2 \\ -13 & 9 \end{pmatrix}
\]

(b) Similarly we have \( B^{-1} = \begin{pmatrix} 5 & -7 \\ -12 & 17 \end{pmatrix} \).

(c) By (6.1) we have

\[
\det \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix} = (5 \times 1) - (3 \times 4) = -7
\]

So using (6.2) we have

\[
C^{-1} = -\frac{1}{7} \begin{pmatrix} 1 & -4 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1/7 & 4/7 \\ 3/7 & -5/7 \end{pmatrix}
\]

(d) We are given the matrix

\[
D = \begin{pmatrix} 3 & -5 & 3 \\ 2 & 1 & -7 \\ -10 & 4 & 5 \end{pmatrix}
\]

What do we need to find?

The inverse matrix \( D^{-1} \) and to find \( D^{-1} \) we have to evaluate the determinant and the adjoint of \( D \).

\[
\det \begin{pmatrix} 3 & -5 & 3 \\ 2 & 1 & -7 \\ -10 & 4 & 5 \end{pmatrix} = 3 \det \begin{pmatrix} 1 & -7 \\ 4 & 5 \end{pmatrix} - (-5) \det \begin{pmatrix} 2 & -7 \\ -10 & 5 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 1 \\ -10 & 4 \end{pmatrix}
\]

\[
= 3[(1 \times 5) - (4 \times (-7))] + 5[(2 \times 5) - (10 \times 7)] + 3[(2 \times 4) - (-10 \times 1)]
\]

\[
= -147
\]

Next we find \( \text{adj}(D) \), which is the cofactor matrix transposed. The cofactor matrix can be obtained using the method described in solution 7. Thus
Complete Solutions to Exercises 6.2

\[
C = \begin{bmatrix}
33 & 60 & 18 \\
37 & 45 & 38 \\
32 & 27 & 13 \\
\end{bmatrix}
\]

Transposing this gives \( adjA \)

\[
adjA = C^T = \begin{bmatrix}
33 & 37 & 32 \\
60 & 45 & 27 \\
18 & 38 & 13 \\
\end{bmatrix}
\]

By (6.13) we have

\[
D^{-1} = -\frac{1}{147} \begin{bmatrix}
33 & 37 & 32 \\
60 & 45 & 27 \\
18 & 38 & 13 \\
\end{bmatrix}
\]

6. (a) Since there are two zeros in the second row it is easier to expand along this row. Remember the place signs start with + and then alternate.

\[
\det \begin{bmatrix}
2 & 3 & 5 \\
0 & 0 & 6 \\
1 & 5 & 3 \\
\end{bmatrix} = -0 \det \begin{bmatrix}
3 & 5 \\
5 & 3 \\
\end{bmatrix} + 0 \det \begin{bmatrix}
2 & 5 \\
1 & 3 \\
\end{bmatrix} - 6 \det \begin{bmatrix}
2 & 3 \\
1 & 5 \\
\end{bmatrix}
= 0 + 0 - 6[(2 \times 5) - (1 \times 3)] = -42
\]

(b) Similarly since there is a zero along the bottom row, expand along this row.

\[
\det \begin{bmatrix}
6 & 7 & 1 \\
1 & 3 & 2 \\
0 & 1 & 5 \\
\end{bmatrix} = 0 \det \begin{bmatrix}
7 & 1 \\
3 & 2 \\
\end{bmatrix} - 1 \det \begin{bmatrix}
6 & 1 \\
1 & 2 \\
\end{bmatrix} + 5 \det \begin{bmatrix}
6 & 7 \\
1 & 3 \\
\end{bmatrix}
= 0 - 1[(6 \times 2) - 1] + 5[(6 \times 3) - (1 \times 7)] = 44
\]

(c) Expand along the first column since it contains two zeros:

\[
\det \begin{bmatrix}
1 & 5 & 1 \\
0 & 3 & 7 \\
0 & 2 & 9 \\
\end{bmatrix} = 1 \det \begin{bmatrix}
3 & 7 \\
2 & 9 \\
\end{bmatrix} - 0 \det \begin{bmatrix}
5 & 1 \\
2 & 9 \\
\end{bmatrix} + 0 \det \begin{bmatrix}
5 & 1 \\
3 & 7 \\
\end{bmatrix}
= 1[(3 \times 9) - (2 \times 7)] - 0 + 0 = 13
\]

(d) Expanding along the second column

\[
\det \begin{bmatrix}
9 & 5 & 1 \\
13 & 0 & 2 \\
11 & 0 & 3 \\
\end{bmatrix} = -5 \det \begin{bmatrix}
13 & 2 \\
11 & 3 \\
\end{bmatrix}
= -5[(13 \times 3) - (11 \times 2)] = -85
\]

7. (a) The triangle given by (0, 0), (3, 2), (7, -4) is illustrated below:
By using the formula in the question we have
\[
\frac{1}{2} \det \begin{pmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 7 & -4 & 1 \end{pmatrix} = \frac{1}{2} \left[ 1 \times \det \begin{pmatrix} 3 & 2 \\ 7 & -4 \end{pmatrix} \right] = \frac{1}{2} (-12 - 14) = -13
\]

Area = $-13 = 13$ units$^2$.

(b) The triangle given by $(-3, 2), (2, 6), (8, -3)$ is illustrated below;

The area is given by
\[
\frac{1}{2} \det \begin{pmatrix} -3 & 2 & 1 \\ 2 & 6 & 1 \\ 8 & -3 & 1 \end{pmatrix} = \frac{1}{2} \left[ -3 \times \det \begin{pmatrix} 6 & 1 \\ -3 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 1 \\ 8 & 1 \end{pmatrix} + \det \begin{pmatrix} 2 & 6 \\ 8 & -3 \end{pmatrix} \right] \\
= \frac{1}{2} \left[ -3(6 + 3) - 2[2 - 8] + [-6 - 48] \right] \\
= \frac{1}{2} \left[ -3(9) - 2[-6] - 54 \right] = \frac{1}{2}(-69) = -34.5
\]

Area = $-34.5 = 34.5$ units$^2$.

(c) We are given the points $(-2, -1), (1, 5)$ and $(0.5, 4)$ which are illustrated below:
What is the area in this case?

Area is zero because we have **no** triangle. Check that
\[
\frac{1}{2} \begin{vmatrix} -2 & -1 & 1 \\ 1 & 5 & 1 \\ 0.5 & 4 & 1 \end{vmatrix} = 0.
\]

All the points lie on a line or we say the three points are **collinear** and we can use the determinate to test if given points are **collinear** (lie on one line). We conclude that if
\[
det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0
\]
then \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are **collinear**.

8. (a) We need to find the equation through the points \((1, 2)\) and \((5, 6)\):
Substituting \(x_1 = 1, y_1 = 2\) and \(x_2 = 5, y_2 = 6\) into the given formula:
\[
det \begin{pmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 1 \end{pmatrix} = -4 - 4x + 4y = 0 \quad \Rightarrow \quad y = x + 1
\]

We can plot this:

(b) Similarly we have to find the equation through the points \((-3, 7)\) and \((10, 10)\):
Substituting \( x_1 = -3, \ y_1 = 7 \) and \( x_2 = 10, \ y_2 = 10 \) into the given formula:

\[
\det \begin{pmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 10 & 10 & 1 \end{pmatrix} = 13y - 3x - 100 = 0 \quad \Rightarrow \quad y = \frac{1}{13}(3x + 100)
\]

We can plot this:

(c) Need to find the equation through the points \((-3, 7)\) and \((9, -21)\):
Substituting \( x_1 = -3, \ y_1 = 7 \) and \( x_2 = 9, \ y_2 = -21 \) into the given formula:

\[
\det \begin{pmatrix} x & y & 1 \\ -3 & 7 & 1 \\ 9 & -21 & 1 \end{pmatrix} = 28x + 12y = 0 \quad \Rightarrow \quad y = -\frac{28}{12}x = -\frac{7}{3}x
\]

We can plot this:

9. The place sign of \( a_{31} \) is \((-1)^{3+1} = (-1)^4 = 1\). The place sign of \( a_{46} \) is

\((-1)^{5+6} = (-1)^{11} = -1\)

The place sign of \( a_{62} \) is \((-1)^{6+2} = (-1)^8 = 1\). The place sign of \( a_{65} \) is \((-1)^{5+6} = -1\).

Since we have a 6 by 6 matrix therefore there is no \( a_{71} \) entry in matrix \( A \).

10. The place sign of \( a_{nm} \) is equal to \((-1)^{m+n} = (-1)^{m+n} \) and \((-1)^{m+n} \) is the place sign of \( a_{nm} \).
Hence we have our result.

\[
\begin{pmatrix} a & b & c & d \\
0 & 0 & 0 & 0 \\
e & f & g & h \\
i & j & k & l
\end{pmatrix}
\]

11. To find \( \det \begin{pmatrix} k & 1 & 2 \\
0 & k & 2 \\
5 & -5 & k
\end{pmatrix} \) we expand along the second row. Why?

Because all the entries along the second row is zero therefore \( \det(A) = 0 \).

12. We need to find the value of \( k \) where \( \det(A) \neq 0 \) [Not Zero]. Easier to find the values of \( k \) where \( \det(A) = 0 \):

\[
\det \begin{pmatrix} k & 1 & 2 \\
0 & 2 & 2 \\
5 & -5 & 2
\end{pmatrix} = 0 + k \det \begin{pmatrix} 2 & 2 \\
5 & k
\end{pmatrix} - 2 \det \begin{pmatrix} 1 & 1 \\
5 & -5
\end{pmatrix}
\]

\[
= k(k^2 - 10) - 2(-5k - 5)
\]

\[
= k^3 - 10k + 10k + 10 = k^3 + 10 = 0
\]

The matrix in invertible for all real values of \( k \) apart from where \( k^3 + 10 = 0 \) or \( k^3 = -10 \).

13. Expanding along the first row:

\[
\det \begin{pmatrix} 1 & 1 & 1 \\
x & y & z \\
x^2 & y^2 & z^2
\end{pmatrix} = \det \begin{pmatrix} y & z \\
x^2 & z^2
\end{pmatrix} - \det \begin{pmatrix} x & z \\
x^2 & z^2
\end{pmatrix} + \det \begin{pmatrix} x & y \\
x^2 & y^2
\end{pmatrix}
\]

\[
= (yz^2 - y^2z) - (xz^2 - x^2z) + (xy^2 - x^2y)
\]

\[
= yz^2 - y^2z - xz^2 + x^2z + xy^2 - x^2y
\]

Expanding the Right-Hand Side of the given result, which is \((x - y)(y - z)(z - x)\), yields

\[
(x - y)(y - z)(z - x) = (xy - xz - y^2 + yz)(z - x)
\]

\[
= xyz - x^2y - xz^2 - y^2z + xy^2 + yz^2 - xyz
\]

\[
= yz^2 - y^2z - xz^2 + x^2z + xy^2 - x^2y
\]

Comparing our answers gives our required result.

14. We need to find the absolute value of the following determinant:

\[
\det \begin{pmatrix} 1 & 2 & 7 \\
2 & 3 & 10 \\
1 & 5 & -1
\end{pmatrix} = -7
\]

The volume is given by \(|-7| = 7\) unit\(^3\).

15. We need to show that the determinant of the rotational matrix \( R \) is equal to 1:

\[
\det \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1
\]
The determinant of one means that the volume of the transformed object has not changed as we would expect when an object has been rotated.

16. We have
\[
\det(J) = \det \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix}
\]
\[
= r \cos^2(\theta) + r \sin^2(\theta) = r \left[ \cos^2(\theta) + \sin^2(\theta) \right] = r(1) = r
\]

17. We are given that
\[
\det \begin{pmatrix} \cos(\theta) \sin(\phi) & -\rho \sin(\theta) \sin(\phi) & \rho \cos(\theta) \cos(\phi) \\ 
\sin(\theta) \sin(\phi) & \rho \cos(\theta) \sin(\phi) & \rho \sin(\theta) \cos(\phi) \\ 
\cos(\phi) & 0 & -\rho \sin(\phi) \end{pmatrix}
\]
\[
= \begin{vmatrix} \cos(\phi) \end{vmatrix} \begin{vmatrix} -\rho \sin(\theta) \sin(\phi) & \rho \cos(\theta) \cos(\phi) \\ 
\rho \cos(\theta) \sin(\phi) & \rho \sin(\theta) \cos(\phi) \end{vmatrix} + \begin{vmatrix} -\rho \sin(\phi) \end{vmatrix} \begin{vmatrix} \cos(\theta) \sin(\phi) & -\rho \sin(\theta) \sin(\phi) \\ 
\sin(\theta) \sin(\phi) & \rho \cos(\theta) \sin(\phi) \end{vmatrix}
\]
\[
= \begin{vmatrix} \cos(\phi) \end{vmatrix} \left[ -\rho^2 \sin^2(\theta) \sin(\phi) \cos(\phi) - \rho^2 \cos^2(\theta) \cos(\phi) \sin(\phi) \right] + \begin{vmatrix} -\rho \sin(\phi) \end{vmatrix} \left[ \rho \cos^2(\theta) \sin^2(\phi) + \rho \sin^2(\theta) \sin^2(\phi) \right]
\]
\[
= -\rho^2 \left[ \cos^2(\phi) \sin(\phi) \left[ \sin^2(\theta) + \cos^2(\theta) \right] + \sin(\phi) \sin^2(\phi) \left[ \sin^2(\theta) + \cos^2(\theta) \right] \right]
\]
\[
= -\rho^2 \sin^2(\phi) \cos^2(\phi) + \sin^2(\phi) \sin^2(\phi) = -\rho^2 \sin(\phi)
\]
Taking the modulus of this gives \( \rho^2 \sin(\phi) \).

18. We have
\[
W(1, \cos(x), \sin(x)) = \det \begin{pmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{pmatrix}
\]
\[
= \sin^2(x) + \cos^2(x) = 1 \quad \text{[Expanding along the first column]}
\]

19. We need to prove for every natural number \( n \) that \( \det(I_n) = 1 \). Remember \( I_n \) is the \( n \times n \) identity matrix. We use proof by induction. \textit{What is the procedure for proof by induction?}

(i) Prove the result for a base case \( n_0 \).
(ii) Assume the result is true for \( n = k \).
(iii) Prove the result for \( n = k + 1 \).

\textit{Proof.}

\textbf{Step (i):}

For \( n = 2 \) we have the identity \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and clearly
Complete Solutions to Exercises 6.2

\[ \det(I_2) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \]

Hence the result is true for \( n = 2 \).

**Step (ii):**
Assume the result is true for \( n = k \) that is \( \det(I_k) = 1 \).

**Step (iii):**
Need to prove it for \( n = k + 1 \) that is required to prove \( \det(I_{k+1}) = 1 \).

\( I_{k+1} \) is the \( k + 1 \) by \( k + 1 \) identity matrix:

\[
I_{k+1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

To find the determinant of \( I_{k+1} \) we can expand along the first row which is 1 times the determinant of the remaining matrix after deleting the first row and column. The remaining matrix is \( I_k \) and by assumption we have \( \det(I_k) = 1 \) therefore

\[
\det(I_{k+1}) = 1 \times 1 = 1
\]

Hence we have proven our result by induction.

\[ \blacksquare \]

20. **Proof.** Since \( A \) is invertible we have \( AA^{-1} = I \) therefore \( \det(AA^{-1}) = \det(I) = 1 \) By Question 19

\[ \blacksquare \]

21. **Proof.**
Expanding along the zero row or column gives

\[
0 \det(\ ) + 0 \det(\ ) + 0 \det(\ ) + \cdots + 0 \det(\ ) = 0
\]

Hence our result.

\[ \blacksquare \]

22. **Proof.**

Let \( A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \) then by expanding along the first row and using

(6.7)

\[
\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}
\]

we have

\[
\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}
\]

Taking the transpose of matrix \( A \) we have \( A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \) Expanding along the first column of the transposed matrix gives
\[ \det(A^T) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{nn}C_{nn} \]

Note that the cofactors are identical because we delete the same elements of the matrix whether we expand along the first row of matrix \( A \) or the first column of the transposed matrix \( A^T \).

Hence \( \det(A^T) = \det(A) \).

\[ \Box \]

23. **Proof.**

Consider the matrix \( B \) obtained from matrix \( A \) by multiplying the \( i \)th row by a scalar \( k \).

\[
\begin{align*}
A &= \begin{pmatrix}
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{ni1} & a_{ni2} & \cdots & a_{nin}
\end{pmatrix} \quad \text{and} \\
B &= \begin{pmatrix}
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
ka_{i1} & ka_{i2} & \cdots & ka_{in}
\end{pmatrix}
\end{align*}
\]

We can find the determinant of matrix \( B \) by expanding along the \( i \)th row and using (6.8):

\[
\det(B) = ka_{i1}C_{i1} + ka_{i2}C_{i2} + ka_{i3}C_{i3} + \cdots + ka_{in}C_{in}
\]

\[
= k\left( a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} \right)
\]

\[
= k\det(A)
\]

Hence we have our required result, that is \( \det(B) = k \det(A) \).

\[ \Box \]

24. Need to prove \( \det(kA) = k^n \det(A) \).

**Proof.**

Let \( A_1 \) be the matrix obtained from \( A \) by multiplying a row by \( k \). Then by result of question 23 we have \( \det(A_1) = k \det(A) \).

Let \( A_2 \) be the matrix obtained from \( A_1 \) by multiplying a non-\( k \) row by \( k \). Then by result of question 23 we have

\[
\det(A_2) = k \det(A_1) = kk \det(A) = k^2 \det(A)
\]

Continuing in this manner we obtain a matrix \( A_n \) from \( A_{n-1} \) by multiplying the last non-\( k \) row by \( k \). Note that to obtain the matrix \( A_n \) we need to multiply each of the \( n \) rows by the scalar \( k \). We have

\[
\det(A_n) = k \det(A_{n-1}) = kk \det(A_{n-2}) = kk \cdots k \det(A) = k^n \det(A)
\]

We have proven that \( \det(kA) = k^n \det(A) \) where \( A \) is a \( n \) by \( n \) matrix.

\[ \Box \]