1. (a) We have

\[ u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Evaluating the inner (dot) product of these gives

\[ \langle u, v \rangle = u \cdot v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1 \cdot (-1)) + (1 \cdot 1) = -1 + 1 = 0 \]

Thus vectors \( u \) and \( v \) are orthogonal.

(b) We have \( u = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \) and \( v = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \):

Evaluating the inner (dot) product of these gives

\[ \langle u, v \rangle = u \cdot v = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix} = (-2 \cdot 3) + (-3 \cdot (-2)) = -6 + 6 = 0 \]

Thus vectors \( u \) and \( v \) are orthogonal.

(c) We are given \( u = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \) and \( v = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \):

Evaluating the inner (dot) product of these gives
\[\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \left( \begin{array}{c} -4 \\ 5 \\ 4 \end{array} \right) \cdot \left( \begin{array}{c} 5 \\ 0 \\ 0 \end{array} \right) = (-4 \times 5) + (5 \times 4) = -20 + 20 = 0\]

Thus vectors \(\mathbf{u}\) and \(\mathbf{v}\) are orthogonal.

(d) We are given \(\mathbf{u} = \left( \begin{array}{c} 2 \\ 7 \end{array} \right)\) and \(\mathbf{v} = \left( \begin{array}{c} -7 \\ 2 \end{array} \right)\):

Evaluating the inner (dot) product of these gives
\[\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \left( \begin{array}{c} 2 \\ 7 \end{array} \right) \cdot \left( \begin{array}{c} -7 \\ 2 \end{array} \right) = (2 \times (-7)) + (7 \times 2) = -14 + 14 = 0\]

Thus vectors \(\mathbf{u}\) and \(\mathbf{v}\) are orthogonal.

2. We need to show that the inner product for \(\mathbf{u} = \left( \begin{array}{c} a \\ b \end{array} \right)\) and \(\mathbf{v} = \left( \begin{array}{c} -b \\ a \end{array} \right)\) is zero.

We have
\[\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \left( \begin{array}{c} a \\ b \end{array} \right) \cdot \left( \begin{array}{c} -b \\ a \end{array} \right) = -ab + ab = 0\]

Hence the vectors \(\mathbf{u}\) and \(\mathbf{v}\) are orthogonal.

3. (a) We are given \(\mathbf{u} = \left( \begin{array}{c} 5 \\ 0 \\ 0 \end{array} \right), \mathbf{v} = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)\) and \(\mathbf{w} = \left( \begin{array}{c} 0 \\ 0 \\ 10 \end{array} \right)\). To check orthogonality:

\[\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \left( \begin{array}{c} 5 \\ 0 \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = (5 \times 0) + (0 \times 1) + (0 \times 0) = 0\]

\[\langle \mathbf{u}, \mathbf{w} \rangle = \mathbf{u} \cdot \mathbf{w} = \left( \begin{array}{c} 5 \\ 0 \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ 0 \\ 10 \end{array} \right) = (5 \times 0) + (0 \times 0) + (0 \times 10) = 0\]

\[\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ 0 \\ 10 \end{array} \right) = (0 \times 0) + (1 \times 0) + (0 \times 10) = 0\]

All 3 vectors \(\mathbf{u}, \mathbf{v}\) and \(\mathbf{w}\) are orthogonal.
(b) We are given the vectors \( u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \), \( v = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \) and \( w = \begin{pmatrix} 10 \\ 0 \\ 7 \end{pmatrix} \). What do you notice about the vector \( u \)?

\( u = O \) [Zero Vector]. What do we know about the orthogonality of the zero vector?

By Proposition (4.10) which says every vector is orthogonal to the zero vector. We only need to check orthogonality of vectors \( v \) and \( w \):

\[
\langle v, w \rangle = v \cdot w = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 0 \\ 7 \end{pmatrix} = (1 \times (-3)) + (1 \times 10) + (-3 \times 7) = 0
\]

4. (i) We are given \( A = \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 & 1 \\ 7 & -12 \end{pmatrix} \). Applying \( \langle A, B \rangle = tr(B^T A) \) gives

\[
\langle A, B \rangle = tr(B^T A) = tr \left[ \begin{pmatrix} 2 & 1 \\ 7 & -12 \end{pmatrix}^T \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \right] = tr \left[ \begin{pmatrix} 2 & 7 \\ 1 & -12 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \right] = tr \left( \begin{pmatrix} 41 & * \\ * & -41 \end{pmatrix} \right) = 41 - 41 = 0
\]

Thus matrices \( A \) and \( B \) are orthogonal.

(ii) Since matrices \( A \) and \( B \) are orthogonal we can apply Pythagoras’s Theorem (4.9):

\[
\|A + B\|^2 = \|A\|^2 + \|B\|^2 \quad (*)
\]

We can evaluate \( \|A\|^2 \) and \( \|B\|^2 \) by the inner products:

\[
\|A\|^2 = \langle A, A \rangle = tr(A^T A) = tr \left[ \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix}^T \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \right] = tr \left[ \begin{pmatrix} 3 & 5 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 5 & 4 \end{pmatrix} \right] = tr \left( \begin{pmatrix} 34 & * \\ * & 65 \end{pmatrix} \right) = 34 + 65 = 99
\]

Similarly

\[
\|B\|^2 = \langle B, B \rangle = tr(B^T B) = tr \left[ \begin{pmatrix} 2 & 1 \\ 7 & -12 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 7 & -12 \end{pmatrix} \right] = tr \left[ \begin{pmatrix} 2 & 7 \\ 1 & -12 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 7 & -12 \end{pmatrix} \right] = tr \left( \begin{pmatrix} 53 & * \\ * & 145 \end{pmatrix} \right) = 53 + 145 = 198
\]
Substituting $\|A\|^2 = 99$ and $\|B\|^2 = 198$ into (*) and taking the square root gives:

$$\|A + B\| = \sqrt{\|A\|^2 + \|B\|^2} = \sqrt{99 + 198} = 17.23 \text{ (2dp)}$$

5. (a) We have

$$\langle u, v \rangle = u \cdot v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ k \\ 5 \end{pmatrix} = (1 \times (-2)) + (2 \times 3) + (3 \times k) + (4 \times 5)$$

$$= -2 + 6 + 3k + 20 = 24 + 3k = 0$$

Solving $24 + 3k = 0$ gives $3k = -24 \Rightarrow k = -8$.

(b) Similarly we have

$$\langle u, v \rangle = u \cdot v = \begin{pmatrix} k \\ -1 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ k \\ k \\ 5 \end{pmatrix} = (k \times 2) + (-1 \times 4) + (k \times k) + (1 \times 5)$$

$$= 2k - 4 + k^2 + 5 = k^2 + 2k + 1 = 0$$

Solving the quadratic gives

$$k^2 + 2k + 1 = 0$$

$$(k + 1)^2 = 0 \Rightarrow k = -1$$

6. Cauchy Schwarz Inequality is $|\langle u, v \rangle| \leq \|u\| \|v\|$.

(a) We need to verify $|\langle f, g \rangle| \leq \|f\| \|g\|$. Evaluating the inner product and the norms:

$$\langle f, g \rangle = \langle x, x-1 \rangle = \int_0^1 x(x-1) \, dx$$

$$= \int_0^1 x^2 - x \, dx \quad \text{[Expanding } x(x-1)\text{]}$$

$$= \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \quad \text{[Integrating]}$$

$$= \left[ \frac{1}{3} - \frac{1}{2} \right] = -\frac{1}{6}$$

$$|\langle f, g \rangle| = \left| -\frac{1}{6} \right| = \frac{1}{6}.$$ 

Initially we determine $\|f\|^2$ and $\|g\|^2$ and then take the square root:
\[ \|f\|^2 = \langle f, f \rangle = \int_0^1 f(x)f(x) \, dx \]
\[ = \int_0^1 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3} \]
\[ \|g\|^2 = \langle g, g \rangle = \int_0^1 g(x)g(x) \, dx \]
\[ = \int_0^1 (x-1)(x-1) \, dx \]
\[ = \left[ \frac{x^3}{3} - \frac{2x^2}{2} + x \right]_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3} \]

Taking square root of \( \|f\|^2 = \frac{1}{3} \) and \( \|g\|^2 = \frac{1}{3} \) gives
\[ \|f\| = \frac{1}{\sqrt{3}} \quad \text{and} \quad \|g\| = \frac{1}{\sqrt{3}} \]

Hence \( \|f\| \|g\| = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{1}{3} \) and this shows Cauchy Schwarz Inequality:
\[ |\langle f, g \rangle| = \frac{1}{6} < \frac{1}{3} = \|f\| \|g\| \]

(b) We need to verify \( |\langle f, g \rangle| \leq \|f\| \|g\| \) for \( f = f(x) = 1 \) and \( g = g(x) = e^x \). Evaluating the inner product and the norms for \( f = 1 \) and \( g = e^x \):
\[ \langle f, g \rangle = \langle 1, e^x \rangle = \int_0^1 e^x \, dx \]
\[ = \left[ e^x \right]_0^1 = \left[ e - 1 \right] = e - 1 = 1.72 \quad (2 \text{ dp}) \]
\[ |\langle f, g \rangle| = 1.72 = 1.72. \]

First we determine \( \|f\|^2 \) and \( \|g\|^2 \) and then take the square root:
\[ \|f\|^2 = \langle f, f \rangle = \int_0^1 1 \, dx = \left[ x \right]_0^1 = 1 \]
\[ \|g\|^2 = \langle g, g \rangle = \int_0^1 e^x e^x \, dx \]
\[ = \int_0^1 e^{2x} \, dx = \left[ \frac{e^{2x}}{2} \right]_0^1 = \frac{e^2 - 1}{2} = 3.19 \quad (2 \text{ dp}) \]

Taking square root gives
\[ \|f\| = 1 = 1 \quad \text{and} \quad \|g\| = \sqrt{3.19} = 1.79 \]

Hence \( \|f\| \|g\| = 1 \times 1.79 = 1.79 \) and this shows Cauchy Schwarz Inequality
\[ |\langle f, g \rangle| = 1.72 < 1.79 = \|f\| \|g\| \]

7. Need to show the following result
Since we have the modulus symbol which measures the absolute value so clearly we have the given expression is $\geq 0$. Using Cauchy Schwarz inequality:

\[(4.6) \quad |\langle u, v \rangle| \leq ||u|| \cdot ||v||\]

Gives

$$\frac{x}{||x||}, \frac{y}{||y||} \leq \frac{||x||}{||x||} \cdot \frac{||y||}{||y||}$$

By Proposition (4.11) we know that $\frac{x}{||x||}$ and $\frac{y}{||y||}$ are unit vectors so they have a norm or length of 1. Hence we have our result:

$$\frac{x}{||x||}, \frac{y}{||y||} \leq \frac{||x||}{||x||} \cdot \frac{||y||}{||y||} = 1 \times 1 = 1$$

8. (a) We have

$$\langle f, g \rangle = \langle \cos(x), \sin(x) \rangle$$

$$= \int_{-\pi}^{\pi} \cos(x)\sin(x) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin(2x) \, dx$$

Because $\cos(x)\sin(x) = \frac{1}{2} \sin(2x)$

$$= -\frac{1}{2} \left[ \frac{\cos(2x)}{2} \right]_{-\pi}^{\pi}$$

Because $\int \sin(kx) \, dx = -\frac{\cos(kx)}{k}$

$$= -\frac{1}{4} \left[ \cos(2\pi) - \cos(-2\pi) \right] = -\frac{1}{4} [1-1] = 0$$

Hence $f$ and $g$ are orthogonal.

(b) We need to show $|\langle f, g \rangle| \leq ||f|| \cdot ||g||$. What is $|\langle f, g \rangle|$ equal to?

By part (a) we know $f$ and $g$ are orthogonal therefore $|\langle f, g \rangle| = 0$. By properties of a norm, Proposition (4.5) part (i), we know $||f|| \geq 0$ and $||g|| \geq 0$ therefore we have Cauchy Schwarz Inequality $|\langle f, g \rangle| \leq ||f|| \cdot ||g||$.

(c) The Minkowski Inequality is $||f + g|| \leq ||f|| + ||g||$. We first find the square of these, that is $||f + g||^2$, $||f||^2$, and $||g||^2$, then take the square root:
\[ \| \mathbf{f} + \mathbf{g} \|^2 = \langle \mathbf{f} + \mathbf{g}, \mathbf{f} + \mathbf{g} \rangle \]
\[ = \int_{-\pi}^{\pi} \left[ f(x) + g(x) \right] \left[ f(x) + g(x) \right] \, dx \]
\[ = \int_{-\pi}^{\pi} \left[ \cos(x) + \sin(x) \right]^2 \, dx \]
\[ = \int_{-\pi}^{\pi} \left[ \cos^2(x) + 2\cos(x)\sin(x) + \sin^2(x) \right] \, dx \]
\[ = \int_{-\pi}^{\pi} \left[ 1 + \sin(2x) \right] \, dx \quad \text{[Because } \cos^2(x) + \sin^2(x) = 1 \text{ and } 2\cos(x)\sin(x) = \sin(2x) \text{]} \]
\[ = \left[ x - \frac{\cos(2x)}{2} \right]_{-\pi}^{\pi} \quad \text{[Because } \int \sin(kx) \, dx = -\frac{\cos(kx)}{k} \text{]} \]
\[ = \left[ \pi - \frac{\cos(2\pi)}{2} \right] - \left[ -\frac{\cos(-2\pi)}{2} \right] \quad \text{[Substituting Limits]} \]
\[ = 2\pi \]

Taking the square root of \( \| \mathbf{f} + \mathbf{g} \|^2 = 2\pi \) gives \( \| \mathbf{f} + \mathbf{g} \| = \sqrt{2\pi} \).

We need to find \( \| \mathbf{f} \|^2 \) and \( \| \mathbf{g} \|^2 \):
\[ \| \mathbf{f} \|^2 = \langle \mathbf{f}, \mathbf{f} \rangle \]
\[ = \int_{-\pi}^{\pi} \cos^2(x) \, dx = \pi \quad \text{By Result} \]

Similarly we have
\[ \| \mathbf{g} \|^2 = \langle \mathbf{g}, \mathbf{g} \rangle \]
\[ = \int_{-\pi}^{\pi} \sin^2(x) \, dx = \pi \quad \text{By Result} \]

Taking the square root of these \( \| \mathbf{f} \|^2 = \pi \) and \( \| \mathbf{g} \|^2 = \pi \) gives \( \| \mathbf{f} \| = \sqrt{\pi} \) and \( \| \mathbf{g} \| = \sqrt{\pi} \) respectively. Adding these
\[ \| \mathbf{f} \| + \| \mathbf{g} \| = \sqrt{\pi} + \sqrt{\pi} = 2\sqrt{\pi} \]

Since \( \| \mathbf{f} + \mathbf{g} \| = \sqrt{2\pi} < 2\sqrt{\pi} = \| \mathbf{f} \| + \| \mathbf{g} \| \), therefore we have Minkowski Inequality.

(d) We normalize these vectors \( \mathbf{f} \) and \( \mathbf{g} \) by \( \frac{\mathbf{f}}{\| \mathbf{f} \|} \) and \( \frac{\mathbf{g}}{\| \mathbf{g} \|} \) with \( \mathbf{f} = \cos(x), \mathbf{g} = \sin(x) \),
\[ \| \mathbf{f} \| = \sqrt{\pi} \text{ and } \| \mathbf{g} \| = \sqrt{\pi} : \]
\[ \frac{\mathbf{f}}{\| \mathbf{f} \|} = \frac{\cos(x)}{\sqrt{\pi}} \text{ and } \frac{\mathbf{g}}{\| \mathbf{g} \|} = \frac{\sin(x)}{\sqrt{\pi}} \]

9. We use the given inner product \( \langle \mathbf{f}, \mathbf{g}_n \rangle = \int_{-\pi}^{\pi} f(x)g_n(t) \, dx \).

First we test \( f(x) = 1 \) and \( g_n(t) = \sin(2nt) \):
\[ \langle f, g_n \rangle = \int_0^\pi 1 \sin(2nt) \, dt \]
\[ = \left[ -\frac{\cos(2nt)}{2n} \right]_0^\pi = -\frac{1}{2n} \left( \cos(2n\pi) - \cos(0) \right) = -\frac{1}{2n}[1 - 1] = 0 \]

Hence \( f \) and \( g \) are orthogonal for \( n \geq 1 \).

Now we test \( g_n(t) = \sin(2nt) \) and \( g_m(t) = \sin(2mt) \) where \( n \neq m \):
\[ \langle \sin(2nt), \sin(2mt) \rangle = \int_0^\pi \sin(2nt) \sin(2mt) \, dt = 0 \]

By Result

Hence the set \( \{1, \sin(2t), \sin(4t), \sin(6t), \ldots\} \) is orthogonal.

We need to normalize these vectors by using Proposition (4.11) which is
\[ \langle f, f \rangle = \int_0^\pi (1 \times 1) \, dt \]
\[ = [t]_0^\pi = \pi \]

What is \( \|f\| \) equal to?

Taking the square root gives \( \|f\| = \sqrt{\pi} \). Similarly we find \( \|g\| \).
\[ \|g\|^2 = \langle g, g \rangle = \int_0^\pi \sin^2(2nt) \, dt = \frac{\pi}{2} \]

Taking the square root gives \( \|g\| = \sqrt{\frac{\pi}{2}} \). Normalizing these vectors gives
\[ \frac{f}{\|f\|} = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad \frac{g}{\|g\|} = \frac{\sin(2nt)}{\sqrt{\pi/2}} \]

Because \( \frac{1}{\sqrt{\pi/2}} = \sqrt{\frac{2}{\pi}} \)

The orthonormal set is \( \left\{ \frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \sin(2t), \sqrt{\frac{2}{\pi}} \sin(4t), \sqrt{\frac{2}{\pi}} \sin(6t), \ldots \right\} \).

10. Clearly we have
\[ \langle A, B \rangle = tr \left( B^T A \right) \]
\[ = tr \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] = tr \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] = tr \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] = 0 + 0 = 0 \]

Similarly we can show that
\[ \langle A, C \rangle = 0, \langle A, D \rangle = 0, \langle B, C \rangle = 0, \langle B, D \rangle = 0 \quad \text{and} \quad \langle C, D \rangle = 0 \]

Hence all the matrices are orthogonal. What else do we need to prove?
Required to show that \( \|A\| = \|B\| = \|C\| = \|D\| = 1 \). What is \( \|A\| \) equal to?

First we find \( \|A\|^2 \) and then take the square root:
\[ \|A\|^2 = \langle A, A \rangle = tr(A^T A) \]
\[ = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 + 0 = 1 \]

Taking the square root gives \( \|A\| = 1 \). Similarly we have
\[ \|B\|^2 = \langle B, B \rangle = tr(B^T B) \]
\[ = tr \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]
\[ = tr \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = tr \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 + 1 = 1 \]

Thus \( \|B\| = 1 \). Similarly we have
\[ \|C\|^2 = \langle C, C \rangle = tr(C^T C) \]
\[ = tr \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]
\[ = tr \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = tr \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 + 0 = 1 \]

Hence \( \|C\| = 1 \) and similarly we can also show that \( \|D\| = 1 \). Since \( \|A\| = \|B\| = \|C\| = \|D\| = 1 \)

therefore the matrices are normalized so the set \( S = \{A, B, C, D\} \) is orthonormal.

11. We need to find \( \langle f(t), z(t) \rangle = \int_0^t f(t)z(t) \, dt \) where \( f(t) = 100 \sin(100t) \) and \( z(t) = \cos(10t) \). Substituting these and using the given hint we have
\[ \langle f(t), z(t) \rangle = \int_0^1 [100 \sin(100t) \cos(10t)] \, dt \]
\[ = 100 \int_0^1 [\sin(100t) \cos(10t)] \, dt \]
\[ = 50 \int_0^1 [\sin(110t) + \sin(90t)] \, dt \]
\[ = 50 \left[ -\frac{\cos(110t)}{110} - \frac{\cos(90t)}{90} \right]_0^1 \]
\[ = -50 \left( \frac{\cos(110) + \cos(90)}{110} \right) \left( \frac{110 + 90}{110 + 90} \right) \]
\[ = -50 \left( \frac{-0.999 + 0.448}{110} \right) \left( \frac{110 + 90}{90} \right) \]
\[ = -50 \left( \frac{-0.551}{110} \right) \left( \frac{200}{90} \right) \]
\[ = -100 \left( -0.551 \cdot \frac{200}{90} \right) \]
\[ = 100 \left( -0.0551 \cdot \frac{200}{90} \right) \]
\[ = 1.713 \]

12. Need to prove Proposition (4.11) which claims:
Every non-zero vector \( \mathbf{w} \) in an inner product space \( V \) can be normalized by setting
\( \mathbf{u} = \frac{\mathbf{w}}{\| \mathbf{w} \|} \).

**Proof.**

Let \( \mathbf{w} \) be a non-zero arbitrary vector in \( V \) and \( \mathbf{u} = \frac{\mathbf{w}}{\| \mathbf{w} \|} \). What do we need to prove?

Required to show that \( \| \mathbf{u} \| = 1 \). We have
\[ \| \mathbf{u} \| = \frac{\| \mathbf{w} \|}{\| \mathbf{w} \|} = \frac{\| \mathbf{w} \|}{\| \mathbf{w} \|} \]
[Writing \( \frac{x}{y} = \frac{1}{y} \cdot x \)]
\[ = \frac{1}{\| \mathbf{w} \|} \| \mathbf{w} \| \]
[By (4.5) part (iii) which is \( \| k \mathbf{u} \| = |k| \| \mathbf{u} \| \)]
\[ = \frac{1}{\| \mathbf{w} \|} \| \mathbf{w} \| = 1 \]
[\( \frac{1}{\| \mathbf{w} \|} > 0 \) therefore \( \frac{1}{\| \mathbf{w} \|} = \frac{1}{\| \mathbf{w} \|} \)]

Since the vector has length of 1 so every **non-zero** vector can be normalized. 

13. **Proof.**

Since \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal so we have
\[ \|u - v\|^2 = \langle u - v, u - v \rangle \]
\[ = \langle u, u \rangle + \langle u, -v \rangle + \langle -v, u \rangle + \langle -v, -v \rangle \]
\[ = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 \]
\[ = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 \quad \text{[Because } u \text{ and } v \text{ are orthogonal]} \]
\[ = \|u\|^2 + \|v\|^2 \]

14. (a) Vectors \( u \) and \( v \) are orthonormal therefore we have \( \langle u, v \rangle = 0, \|u\| = 1 \) and \( \|v\| = 1 \). To find \( \|u + v\| \) we first determine \( \|u + v\|^2 \) and then take the square root. We can use Pythagoras Theorem (4.9) because \( u \) and \( v \) are orthogonal:
\[ \|u + v\|^2 = \|u\|^2 + \|v\|^2 = 1 + 1 = 2 \]
Thus taking the square root of both sides gives \( \|u + v\| = \sqrt{2} \).
(b) By using the result of question 13 above we have \( \|u - v\| = \sqrt{2} \).

15. (i) We need to prove \( \|u_1 + u_2 + u_3 + \cdots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \cdots + \|u_n\|^2 \):

**How?**
By using mathematical induction.

**Proof.**
Clearly the result holds for \( n = 2 \) because by (4.9) we have
\[ \|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2 \]
Assume the result is true for \( n = k \), that is
\[ \|u_1 + u_2 + u_3 + \cdots + u_k\|^2 = \|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \cdots + \|u_k\|^2 \quad (*) \]
We need to prove the result for \( n = k + 1 \), that is we are required to prove
\[ \|u_1 + u_2 + u_3 + \cdots + u_k + u_{k+1}\|^2 = \|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \cdots + \|u_k\|^2 + \|u_{k+1}\|^2 \]
Examining the Left Hand Side we have
\[ \|u_1 + u_2 + u_3 + \cdots + u_k + u_{k+1}\|^2 = \|u_1 + u_2 + u_3 + \cdots + u_k + u_{k+1}\|^2 \]
\[ = \|u_1 + u_2 + u_3 + \cdots + u_k\|^2 + \|u_{k+1}\|^2 \quad \text{[By (4.9)]} \]
\[ = \|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \cdots + \|u_k\|^2 + \|u_{k+1}\|^2 \quad \text{By (*)} \]
This completes the proof.

(ii) We are given that \( \{f_1, f_2, f_3, \ldots, f_n\} \) be an orthonormal set of vectors in \( C[0, \pi] \).
Since \( \{f_1, f_2, f_3, \ldots, f_n\} \) is an orthonormal set so the vectors are orthogonal. Applying Pythagoras (4.9) we have
\[ \|f_1 + f_2 + f_3 + \cdots + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2 + \cdots + \|f_n\|^2 \]
Each of these vectors are normalized so they have a length of 1. Hence
\[ \| f_1 + f_2 + f_3 + \ldots + f_n \|^2 = \| f_1 \|^2 + \| f_2 \|^2 + \| f_3 \|^2 + \ldots + \| f_n \|^2 \]
\[ = 1 + 1 + 1 + \ldots + 1 = n \]

Taking the square root gives \[ \| f_1 + f_2 + f_3 + \ldots + f_n \| = \sqrt{n}. \]

16. We need to prove that the vectors \( \{ u_1, u_2, u_3, \ldots, u_n \} \) are orthogonal \( \iff \) \( \{ k_1 u_1, k_2 u_2, k_3 u_3, \ldots, k_n u_n \} \) are orthogonal.

**Proof.**

(\( \Rightarrow \)) Let \( u_i \) and \( u_j \) be arbitrary vectors where \( i \neq j \). Consider the inner product of

\[ \langle k_i u_i, k_j u_j \rangle = k_i k_j \langle u_i, u_j \rangle \]

[Because by (4.1) (iii) \( \langle k u, v \rangle = k \langle u, v \rangle \)]

\[ = k_i k_j \langle u_i, u_j \rangle = 0 \]

[Because \( u_i \) and \( u_j \) are orthogonal]

Hence the arbitrary vectors \( k_i u_i \) and \( k_j u_j \) are orthogonal because their inner product is zero. Since \( k_i u_i \) and \( k_j u_j \) were arbitrary vectors so

\[ \{ k_1 u_1, k_2 u_2, k_3 u_3, \ldots, k_n u_n \} \]

is an orthogonal set of vectors.

(\( \Leftarrow \)). We can assume \( \{ k_1 u_1, k_2 u_2, k_3 u_3, \ldots, k_n u_n \} \) are a set of orthogonal vectors. Consider two arbitrary vectors in this set \( k_i u_i \) and \( k_j u_j \) where \( i \neq j \). We have

\[ \langle k_i u_i, k_j u_j \rangle = 0. \]

Expanding this

\[ \langle k_i u_i, k_j u_j \rangle = k_i k_j \langle u_i, u_j \rangle = 0 \]

We are given that the scalars \( k \) are non-zero therefore

\[ k_i k_j \langle u_i, u_j \rangle = 0 \Rightarrow \langle u_i, u_j \rangle = 0 \Rightarrow u_i \text{ and } u_j \text{ are orthogonal} \]

This completes our proof.

\[ \blacksquare \]

If one of the scalars is zero, \( k_j = 0 \) say. We have \( k_j u_j = \mathbf{0} \) which means that

\( k_j u_j \) is orthogonal to all the vectors in \( \{ k_1 u_1, k_2 u_2, k_3 u_3, \ldots, k_n u_n \} \)

However the inner product of this vector \( k_j u_j = \mathbf{0} \) and \( k_i u_i \) is zero, that is

\[ \langle k_i u_i, k_j u_j \rangle = k_i k_j \langle u_i, u_j \rangle = 0 \text{ because } k_j = 0 \]

But \( \langle u_i, u_j \rangle \neq 0 \) which means the vectors \( u_i \) and \( u_j \) are not orthogonal. Hence if one of the scalars is zero then the orthogonal set

\[ \{ k_1 u_1, k_2 u_2, k_3 u_3, \ldots, k_n u_n \} \Rightarrow \{ u_1, u_2, u_3, \ldots, u_n \} \]

are orthogonal

17. We are required to prove \( \| u + v \| \leq \| u \| + \| v \| \).

**Proof.**
\[ \|u + v\|^2 = \langle u + v, u + v \rangle \]
\[ = \langle u, u + v \rangle + \langle v, u + v \rangle \]
\[ = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \]
\[ = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2 \]
\[ \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{[Because } x \leq |x|] \]
\[ \leq \|u\|^2 + 2\|u\||\|v\| + \|v\|^2 \quad \text{[Using CSI (5.6) } \langle u, v \rangle \leq \|u\||\|v\| \text{ ]} \]
\[ = (\|u\| + \|v\|)^2 \]

We have
\[ \|u + v\|^2 \leq (\|u\| + \|v\|)^2 \]
\[ \|u + v\| \leq \|u\| + \|v\| \quad \text{[Taking Square Roots]} \]

This was our required inequality.