Complete Solutions to Exercises 1.1

1. (a) \( x - y - z = 3 \) is linear because the index of \( x, y \) and \( z \) is one.
(b) \( \sqrt{x} + y + z = 6 \) is not linear because \( \sqrt{x} = x^{1/2} \) that is the index of \( x \) is \( 1/2 \).
(c) \( \cos(x) + \sin(y) = 1 \) is not linear because \( x \) and \( y \) are arguments of trigonometric functions.
(d) \( e^{x+y+z} = 1 \) is not linear because \( x, y \) and \( z \) are arguments of the exponential function.
(e) \( x - 2y + 5z = \sqrt{3} \) is linear because the index of \( x, y \) and \( z \) is one.
(f) \( x = -3y \) is linear because the index of \( x \) and \( y \) is one.
(g) \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) is linear because the place-holder \( x \) has index 1 and the Right Hand Side, \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), is a constant.
(h) \( \pi x + y + ez = 5 \) is linear because the index of \( x, y \) and \( z \) is one.
(i) \( \sqrt{2x} + \frac{1}{2} y + z = 0 \) is linear because the index of \( x, y \) and \( z \) is one.
(j) \( \sinh^{-1}(x) = \ln|x + \sqrt{x^2 + 1}| \) is not linear because \( x \) is an argument of a hyperbolic and logarithmic functions.
(k) \( \frac{\pi}{2} x - \sqrt{2} y + z \sin(\pi) = 0 \) is linear because the index of \( x, y \) and \( z \) is one.
(l) \( x^0 + y^0 + z^0 = 0 \) is linear because \( 2^0 = 3^0 = 1 \) so the index of each unknown is 1.
(m) \( y^{\cos(x) + \sin(x)} + x - z = 9 \) is linear because the index of \( x, y \) and \( z \) is one. Remember from trigonometry that \( \cos^2(x) + \sin^2(x) = 1 \).
(n) Remember from your work on complex numbers that \( e^{i\pi} = -1 \) therefore \( x^i = x^{-1} = 6 \) is not linear because the index of \( x \) is \( -1 \).

2. (a) We add the two given simultaneous equations:
\[
x + y = 2 \\
\begin{aligned}
x - y &= 0 \\
2x + 0 &= 2
\end{aligned}
\]
We have \( 2x = 2 \) which gives \( x = 1 \). How do we find the value of the other unknown, \( y \)?
Substitute \( x = 1 \) into the first equation \( x + y = 2 \):
\[
1 + y = 2 \text{ which gives } y = 1
\]
Hence \( x = 1 \) and \( y = 1 \) is the solution.
(b) We can label each linear equation
\[
\begin{align*}
2x - 3y &= 5 & (*)& \\
x - y &= 2 & (**)
\end{align*}
\]
How do we eliminate one of the unknowns?
Eliminate \( x \) by multiplying equation (**) by 2 and subtracting from equation (*):
\[2x - 3y = 5 \quad (\ast)\]
\[- (2x - 2y = 4) \quad \text{[Multiplying (\ast\ast) by 2]}\]
\[0 - y = 1\]

Hence \(y = -1\). \textit{How do we determine }\(x\)?

By substituting \(y = -1\) into the given equation \(x - y = 2 \quad (\ast\ast)\):
\[
\begin{align*}
x - (-1) &= 2 \\
x + 1 &= 2 \quad \text{which gives } x = 1
\end{align*}
\]
The solution is \(x = 1\) and \(y = -1\).

(c) We label each equation:
\[
\begin{align*}
2x - 3y &= 35 \quad (\Box) \\
x - y &= 2 \quad (\Box\Box)
\end{align*}
\]

We need to eliminate one of the unknowns. \textit{Which one?}

To make life easier it is better to eliminate \(x\). \textit{How?}

Multiply equation \((\Box)\) by 2 and subtract from equation \((\Box\Box)\):
\[
\begin{align*}
2x - 3y &= 35 \quad (\Box) \\
- (2x - 2y = 4) \quad \text{[Multiplying (\Box\Box) by 2]}\)
\[
0 - y = 31
\]

We have \(y = -31\). \textit{How do we find }\(x\)?

Substitute \(y = -31\) into the given equation \(x - y = 2 \quad (\Box\Box)\):
\[
\begin{align*}
x - (-31) &= 2 \\
x + 31 &= 2 \quad \text{which gives } x = -29
\end{align*}
\]
The solution is \(x = -29\) and \(y = -31\).

(d) We label each equation:
\[
\begin{align*}
5x - 7y &= 2 \quad (\dagger) \\
9x - 3y &= 6 \quad (\dagger\dagger)
\end{align*}
\]

We need to eliminate one of the unknowns. \textit{Which one?}

Eliminate \(y\). \textit{How?}

Multiply equation \((\dagger)\) by 3 and multiply \((\dagger\dagger)\) by 7:
\[
\begin{align*}
15x - 21y &= 6 \quad \text{[Multiplying (\dagger) by 3]} \\
63x - 21y &= 42 \quad \text{[Multiplying (\dagger\dagger) by 7]}\]
\]

To eliminate \(y\) we subtract these equations
\[
\begin{align*}
63x - 21y &= 42 \\
- (15x - 21y = 6) \quad \text{[Subtracting]}\)
\[
48x - 0 = 36
\]

From \(48x = 36\) we have \(x = \frac{36}{48} = \frac{3}{4}\). \textit{How do we find }\(y\)?

Substitute \(x = \frac{3}{4}\) into the given equation \(5x - 7y = 2 \quad (\dagger)\):
\[
5 \left( \frac{3}{4} \right) - 7y = 2 \\
\frac{15}{4} - 7y = 2 \\
7y = \frac{15}{4} - 2 = \frac{7}{4}
\]

How do find \( y \) from \( 7y = \frac{7}{4} \)?
Dividing both sides by 7:
\[
y = \frac{7}{4(7)} = \frac{1}{4} \quad [\text{ Cancelling }]
\]
The solution is \( x = \frac{3}{4} \) and \( y = \frac{1}{4} \).

(e) We are given the equations
\[
\pi x - 5y = 2 \quad (*) \\
\pi x - y = 1 \quad (**) 
\]
Subtracting these equations we have
\[
\pi x - 5y = 2 \\
- \quad \pi x - y = 1 \\
0 - 4y = 1
\]
From the last line \(-4y = 1\) we have \( y = -\frac{1}{4} \). How do we determine \( x \)?
Substitute \( y = -\frac{1}{4} \) into \((**):\)
\[
\pi x \left( -\frac{1}{4} \right) = 1 \\
\pi x + \frac{1}{4} = 1 \quad \Rightarrow \quad \pi x = \frac{3}{4} \quad \text{which gives} \quad x = \frac{3}{4\pi}
\]
The solution is \( x = \frac{3}{4\pi} \) and \( y = -\frac{1}{4} \).

(f) We add the given equations
\[
ex - ey = 2 \\
+ \quad ex + ey = 0 \\
2ex + 0 = 2
\]
Transposing \( 2ex = 2 \) gives \( x = \frac{1}{e} \). What else do we need to find?
The value of \( y \). How?
By substituting \( x = \frac{1}{e} \) into the given second equation \( ex + ey = 0 \):
\[ e \left( \frac{1}{e} \right) + ey = 0 \]
\[ 1 + ey = 0 \]
\[ ey = -1 \quad \text{we have } y = -\frac{1}{e} \]

The solution is \( x = \frac{1}{e} \) and \( y = -\frac{1}{e} \).

3. (a) We can label the given equations:

\[
\begin{align*}
  x + y + z &= 3 \quad (*) \\
  x - y - z &= -1 \quad (**) \\
  2x + y + 5z &= 8 \quad (***)
\end{align*}
\]

What do we need to find?
The values of \( x, y \) and \( z \) that satisfy the linear equations (*) , (**) and (***) . How?

By elimination. If we add the given equations (*) and (**) we get:

\[
\begin{align*}
  x + y + z &= 3 \\
  + \quad x - y - z &= -1 \\
  \therefore 2x + 0 + 0 &= 2
\end{align*}
\]

From the last line \( 2x = 2 \) we have \( x = 1 \) . What else do we need to find?
The values of the place-holders \( y \) and \( z \) . How?

By substituting \( x = 1 \) into the given equations (*) and (***):

\[
\begin{align*}
  1 + y + z &= 3 \\
  2 + y + 5z &= 8
\end{align*}
\]

We can subtract these equations, that is \((\square)-(\square):\)

\[
\begin{align*}
  2 + y + 5z &= 8 \\
  - (1 + y + z) &= 3
\end{align*}
\]

\[
\begin{align*}
  1 + 0 + 4z &= 5
\end{align*}
\]

From the last line we have \( 1 + 4z = 5 \) . Transposing gives

\[ 4z = 4 \quad \text{implies that } z = 1 \]

So far we have \( x = 1 \) and \( z = 1 \) . How can we find \( y \) ?

Substitute these values \( x = 1 \) and \( z = 1 \) into the given equation \( x + y + z = 3 \) \( (*) \).

\[
\begin{align*}
  1 + y + 1 &= 3 \\
  y &= 1
\end{align*}
\]

We have the solution \( x = 1, \ y = 1 \) and \( z = 1 \).

(b) We can label the given equations:

\[
\begin{align*}
  x + 2y - 2z &= 6 \quad (\diamond) \\
  2x - 3y + z &= -10 \quad (\diamond\diamond) \\
  3x - y + 3z &= -16 \quad (\diamond\diamond\diamond)
\end{align*}
\]

What do we need to find?
The values of \( x, y \) and \( z \) that satisfy the given linear equations \( (\diamond) \), \( (\diamond\diamond) \) and \( (\diamond\diamond\diamond) \). How?
By elimination. If we multiply \((\diamond)\) by 2 and then subtract \((\diamond\diamond)\) we get:

\[
2x + 4y - 4z = 12 \quad \text{[Multiplying \((\diamond)\) by 2]}
\]

\[
- (2x - 3y + z = -10) \quad \text{(\diamond\diamond)}
\]

\[
0 + 7y - 5z = 22
\]

We need another linear equation involving \(y\) and \(z\) so that we have two simultaneous equations with unknowns \(y\) and \(z\). *How can we get such an equation?*

By multiplying equation \((\diamond)\) by 3 and then subtracting equation \((\diamond\diamond)\):

\[
3x + 6y - 6z = 18 \quad \text{[Multiplying \((\diamond)\) by 3]}
\]

\[
- (3x - y + 3z = -16) \quad \text{(\diamond\diamond\diamond)}
\]

\[
0 + 7y - 9z = 34
\]

Hence we have obtained two simultaneous linear equations with unknowns \(y\) and \(z\) only:

\[
7y - 5z = 22 \quad \text{(\dagger)}
\]

\[
7y - 9z = 34 \quad \text{(\dagger\dagger)}
\]

Subtracting these equations, \((\dagger) - (\dagger\dagger)\), we have

\[
7y - 5z = 22
\]

\[
- (7y - 9z = 34)
\]

\[
0 + 4z = -12
\]

From the last line \(4z = -12\) we have \(z = -3\). *What else do we need to find?*

The values of \(y\) and \(x\). *How?*

By substituting \(z = -3\) into equation \((\dagger)\):

\[
7y - 5(-3) = 22
\]

\[
7y + 15 = 22
\]

\[
7y = 7 \text{ which gives } y = 1
\]

We have found \(y = 1\) and \(z = -3\). *How can we find \(x\)?*

Substitute these values \(y = 1\) and \(z = -3\) into the equation \(x + 2y - 2z = 6 \quad (\dagger)\).

\[
x + 2(1) - 2(-3) = 6
\]

\[
x + 2 + 6 = 6
\]

\[
x = -2
\]

We have the solution \(x = -2\), \(y = 1\) and \(z = -3\) that satisfy the given linear system.

(c) We can label the given equations:

\[
3x + y - 2z = 4 \quad \text{(\bullet)}
\]

\[
5x - 3y + 10z = 32 \quad \text{(\bullet\bullet)}
\]

\[
7x + 4y + 16z = 13 \quad \text{(\bullet\bullet\bullet)}
\]

*What do we need to find?*

The values of \(x\), \(y\) and \(z\) that satisfy all the linear equations \((\bullet)\), \((\bullet\bullet)\) and \((\bullet\bullet\bullet)\). *How?*

By elimination. If we multiply equation \((\bullet)\) by 3 and then add equation \((\bullet\bullet)\) we get:
9x + 3y - 6z = 12 \ [\text{Multiplying (\bullet) by 3}] \\
+ \quad 5x - 3y + 10z = 32 \quad (\bullet\bullet) \\
14x + 0 + 4z = 44

We need another linear equation involving \( x \) and \( z \) so that we have two simultaneous equations with unknowns \( x \) and \( z \) only. \textit{How can we get such an equation?}

By multiplying the given equation (\bullet) by 4 and then subtract equation (\bullet\bullet):
\[
12x + 4y - 8z = 16 \quad \text{[Multiplying (\bullet) by 4]} \\
- \quad (7x + 4y + 16z = 13) \quad (\bullet\bullet\bullet)
\]
\[
5x + 0 - 24z = 3 \quad \text{[Subtracting]}
\]

Hence we have two simultaneous linear equations with unknowns \( x \) and \( z \) only:
\[
14x + 4z = 44 \quad (\dagger) \\
5x - 24z = 3 \quad (\dagger\dagger)
\]

\textit{How can we determine \( x \) or \( z \)?}

By multiplying (\dagger) by 6 and then adding (\dagger\dagger):
\[
84x + 24z = 264 \quad \text{[Multiplying (\dagger) by 6]} \\
+ \quad 5x - 24z = 3 \quad (\dagger\dagger)
\]
\[
89x - 0 = 267
\]

From the last line \( 89x = 267 \) we have \( x = \frac{267}{89} = 3 \). \textit{What else do we need to find?}

The values of \( y \) and \( z \). \textit{How?}

By substituting \( x = 3 \) into equation \( 14x + 4z = 44 \) (\dagger):
\[
14(3) + 4z = 44 \\
42 + 4z = 44 \\
4z = 44 - 42 = 2 \\
z = \frac{2}{4} = \frac{1}{2}
\]

So far we have found \( x = 3 \) and \( z = \frac{1}{2} \). \textit{How can we find \( y \)?}

Substitute these values \( x = 3 \) and \( z = \frac{1}{2} \) into the given equation \( 3x + y - 2z = 4 \) (\bullet).
\[
3(3) + y - 2\left(\frac{1}{2}\right) = 4 \\
9 + y - 1 = 4 \\
y + 8 = 4 \quad \text{which gives} \quad y = -4
\]

We have the solution \( x = 3, \ y = -4 \) and \( z = \frac{1}{2} \).

(d) We can label the given equations:
\[
6x - 3y + 2z = 31 \quad (*) \\
5x + y + 12z = 36 \quad (**) \\
8x + 5y + z = 11 \quad (***)
\]
What do we need to find?
The values of \(x, y\) and \(z\) that satisfy all the linear equations (*), (**), and (***) How?
By elimination. If we multiply equation (**) by 3 and add equation (*) we get:
\[
6x - 3y + 2z = 31 \quad (*)
\]
\[
15x + 3y + 36z = 108 \quad [\text{Multiplying (** by 3}]
\]
\[
21x + 0 + 38z = 139
\]
We need another linear equation involving \(x\) and \(z\) so that we have two simultaneous equations with two unknowns \(x\) and \(z\) only. How can we get such an equation?
By multiplying the given equation (***) by 5 and then subtracting equation (**):
\[
25x + 5y + 60z = 180 \quad [\text{Multiplying (** by 5}]
\]
\[
8x + 5y + z = 11 \quad (***)
\]
\[
17x + 0 + 59z = 169
\]
Hence we have two simultaneous linear equations with two unknowns \(x\) and \(z\) only:
\[
21x + 38z = 139
\]
\[
17x + 59z = 169
\]
How can we determine \(x\) or \(z\)?
By multiplying the first equation, \(21x + 38z = 139\), by 17
\[
(21 \times 17)x + (38 \times 17)z = (139 \times 17)
\]
\[
357x + 646z = 2363 \quad (†)
\]
and multiplying the second equation, \(17x + 59z = 169\), by 21
\[
(17 \times 21)x + (59 \times 21)z = (169 \times 21)
\]
\[
357x + 1239z = 3549 \quad (‡†)
\]
How can we eliminate \(x\) from these two equations, (†) and (‡†)?
By subtracting:
\[
357x + 1239z = 3549 \quad (‡†)
\]
\[
- (357x + 646z = 2363) \quad (†)
\]
\[
0 + 593z = 1186
\]
From the last line 593\(z = 1186\) we have \(z = \frac{1186}{593} = 2\). What do we need to find?
The values of \(x\) and \(y\). How?
By substituting \(z = 2\) into \(357x + 646z = 2363\) (†):
\[
357x + 646(2) = 2363
\]
\[
357x + 1292 = 2363
\]
\[
357x = 2363 - 1292 = 1071
\]
\[
x = \frac{1071}{357} = 3
\]
We have found \(x = 3\) and \(z = 2\). How can we find \(y\)?
Substitute these values \(x = 3\) and \(z = 2\) into the given equation \(6x - 3y + 2z = 31\) (*).
Complete Solutions to Exercises 1.1

\[ 6(3) - 3y + 2(2) = 31 \]
\[ 18 - 3y + 4 = 31 \]
\[ 22 - 3y = 31 \]
\[ -3y = 31 - 22 = 9 \]
\[ y = \frac{-9}{3} = -3 \]

We have the solution \( x = 3, \ y = -3 \) and \( z = 2 \).

4. The graphs are as follows:
   (a) One Solution
   ![Graph](image1)
   (b) Infinite number of solutions.
   ![Graph](image2)
   (c) No solution.
   ![Graph](image3)
(d) No solution.

(e) No solution.

(f) Unique solution.
5. (a) If we had the given equations:

\[ 7x + y = 10 \]
\[ x - y = 7 \]

we will find a value of \( x \). This means the solution is unique.

(b) What do you notice about the given equations?

\[ 12x + 4y = 16 \]
\[ 8x + 4y = 16 \]

Different coefficients of \( x \) but the same \( y \) give 16. Let \( x = 0 \) then we have the same equation, \( 4y = 16 \). 

(c) What do you notice about these equations:

\[ 2x - y - z = 3 \]
\[ 4x - 2y - 2z = 3 \]

Different coefficients of \( x \), \( y \) and \( z \) give the same answer, 3, therefore the system has no solution.