26.1 Partitions and diagrams

26.1.1
Write down the diagrams representing the following partitions

(i) \([1^23^25^17^1]\);

(ii) \([2^14^26^17^1]\).

Solution

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times \\
(i) & \times & \times & \times \\
\times & \times & \times \\
\times \\
\end{array}
\]

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
(ii) & \times & \times & \times & \times & \times \\
\times & \times & \times \\
\times \\
\end{array}
\]
26.2 Conjugate partitions

26.2.1

Write down the conjugates of the following partitions in standard notation:

(i) \([1^23 5 6]\);

(ii) \([2^2 3^3 5 8]\).

Solution

(i) The diagram for \([1^2 3 5 6]\) is

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \\
\times & \\
\end{array}
\]

Reading down the columns, the conjugate is \([1^2 2^2 3 5]\).

(ii) Similarly the diagram for \([2^2 3^3 5 8]\) is

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times \\
\times & \times & \times \\
\times & \times & \\
\times & \\
\times & \\
\end{array}
\]

and the conjugate is \([1^3 2^2 5 7^2]\).
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26.2.2
Which of the following partitions are self-conjugate?
(i) \([1^2 2^4]\);
(ii) \([1^3 4^5 6]\);
(iii) \([2^2 3^5 2]\);
(iv) \([1^4 2^3 4^8]\).

Solution  The quickest method is to draw the diagram in each case.
(i) 
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\times & \times \\
\times 
\end{array}
No.

(ii) 
\[
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times \\
\times \\
\times \\
\times 
\end{array}
No.

(iii) 
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times \\
\times & \times \\
\times 
\end{array}
Yes.

(iv) 
\[
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times \\
\times & \times \\
\times \\
\times \\
\times \\
\times 
\end{array}
Yes.
26.3 Partitions and generating functions

26.3.1
Write down the generating functions for the sequences whose $n$th terms are

(i) the number of partitions of $n$ into parts equal to 3 or 5;
(ii) the number of partitions of $n$ into parts equal to 2, 4, or 6.

Solution

(i) \[ \frac{1}{(1 - x^3)(1 - x^5)} \]

(ii) \[ \frac{1}{(1 - x^2)(1 - x^4)(1 - x^6)} \]
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26.3.3

In how many ways can one pound (100p) be exchanged for coins of the values 50p, 20p, 10p, and 5p?

Solution  We need the coefficient of $y^{10}$ in

$$(1 - y)^{-1}(1 - y^2)^{-1}(1 - y^4)^{-1}(1 - y^{10})^{-1}.$$  

Following the method in the text,

$$(1 - y^4)^{-1}(1 - y^{10})^{-1} = (1 + y^4 + y^8 + y^{12} + y^{16} + \cdots)(1 + y^{10} + y^{20} + \cdots) = (1 + y^4 + y^8 + y^{10} + y^{12} + y^{14} + y^{16} + y^{18} + 2y^{20} + \cdots).$$

Multiplying by $(1 - y^2)^{-1}$ with detached coefficients

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
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<td>01010</td>
<td>01010</td>
<td>2</td>
</tr>
<tr>
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<td>01010</td>
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<td>1</td>
</tr>
<tr>
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<td>00101</td>
<td>01010</td>
<td>1</td>
<td></td>
</tr>
<tr>
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<td></td>
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<tr>
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<td></td>
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<td>1</td>
<td>00010</td>
<td>0</td>
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<tr>
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<td>0</td>
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<td></td>
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</tr>
<tr>
<td>1</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Multiplying by $(1 - y)^{-1}$ is equivalent to adding the coefficients, so the answer is 49.
26.4 Generating functions for restricted partitions

26.4.1
Write down formulae for the generating functions for the sequences whose \( n \)th terms are

(i) \( p(n \mid \text{each part occurs at most twice}) \);

(ii) \( p(n \mid \text{each part is power of 2}) \);

(iii) \( p(n \mid \text{the smallest part is 5}) \).

Solution

(i) Since any part \( i \) can occur either 0, 1, or 2 times, the generating function is the product of all the terms \( 1 + x^i + x^{2i} \), that is

\[
(1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6) \cdots.
\]

(ii) The parts can be 1, 2, 4, 8, \ldots, each repeated any number of times, so the generating function is

\[
(1 - x)^{-1} (1 - x^2)^{-1} (1 - x^4)^{-1} (1 - x^8)^{-1} \cdots.
\]

(iii) There is at least one part equal to 5, so the generating function is

\[
x^5 (1 - x^5)^{-1} (1 - x^6)^{-1} (1 - x^7)^{-1} \cdots.
\]
26.4.2

Use the method of generating functions to find the number of partitions of 16 in which each part is an odd prime.

**Solution**  
The odd primes less than 16 are 3,5,7,11,13. The generating function for partitions with these parts is

\[(1 - x^3)^{-1} (1 - x^5)^{-1} (1 - x^7)^{-1} (1 - x^{11})^{-1} (1 - x^{13})^{-1} .\]

We have to find the coefficient of \(x^{16}\), so only the terms up to \(x^{16}\) in each factor are required:

\[
\begin{align*}
(1 + x^3 + x^6 + x^9 + x^{12} + x^{15}) &\times (1 + x^5 + x^{10} + x^{15}) \\
(1 + x^7 + x^{11} + x^{15}) &\times (1 + x^{11}) \\
(1 + x^{13} + x^{14} + \cdots) &\times (1 + x^3 + x^6 + x^9 + x^{12} + x^{15}) \\
(1 + x^7 + x^{11} + x^{13} + x^{14} + \cdots) &\times (1 + x^3 + x^6 + x^9 + x^{12} + x^{15}) \\
(1 + x^{11} + x^{12} + x^{13} + x^{14} + \cdots) &\times (1 + x^5 + x^7 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + \cdots)
\end{align*}
\]

Here we can see that there are four terms which contribute 1 to the coefficient of \(x^{16}\), so the number of partitions of the required type is 4. The partitions are

\[13 + 3, \quad 11 + 5, \quad 7 + 3 + 3 + 3, \quad 5 + 5 + 3 + 3.\]
26.4.4

Show that

\[(1 - x)(1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^m}) = 1 - x^{2^{m+1}},\]

and derive the formula

\[(1 - x)^{-1} = (1 + x)(1 + x^2)(1 + x^4) \cdots ,\]

where the right-hand side is the product of the factors \(1 + x^{2^r}\) for all \(r \geq 0\). Deduce that every positive integer has a unique partition whose parts are distinct powers of 2.

Solution

Using the formula \((1 - y)(1 + y) = 1 - y^2\) repeatedly we have

\[
\begin{align*}
(1 - x)(1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^m}) & = (1 - x^2)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^m}) \\
& = (1 - x^4)(1 + x^4) \cdots (1 + x^{2^m}) \\
& = \cdots \\
& = 1 - (x^{2^m})^2 = 1 - x^{2^{m+1}}.
\end{align*}
\]

If \(F(x)\) denotes the infinite product \((1 + x)(1 + x^2)(1 + x^4) \cdots\), the above argument shows that \((1 - x)F(x) = 1\), that is

\[(1 - x)^{-1} = (1 + x)(1 + x^2)(1 + x^4) \cdots .\]

Finally, observe that \(F(x)\) is the generating function for the numbers \(f(n)\), where \(f(n)\) is the number of partitions of \(n\) into distinct parts, all of which are powers of 2. Since \(F(x) = (1 - x)^{-1}\), it follows that \(f(n) = 1\) for all \(n\).
26.5 A mysterious identity

26.5.1
Show that the coefficient of $x^n$ in the power series expansion of

$$(1 + x) (1 + x^2) (1 + x^3) \cdots$$

is even unless $n$ is a number of the form $\frac{1}{2}m (3m \pm 1)$.

Solution
The coefficient of $x^n$ is the number of partitions of $n$ into distinct parts. In the notation used in the text, this is

$$a_n + e_n = 2a_n + (e_n - a_n) = 2a_n + q_n.$$ 

By Theorem 26.5, $q_n$ is zero unless $n = \frac{1}{2}m (3m \pm 1)$. 

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26.5.2

Let \( q_{n,d} \) denote the number of partitions of \( n \) which have \( d \) parts, all distinct. Show that
\[
q_{n,d} = q_{n-d,d} + q_{n-d,d-1} \quad (1 \leq d \leq n).
\]

Solution

We can split the relevant set of partitions as follows.

\[
q_{n,d} = \Big( \text{number in which 1 is not a part} \Big) + \Big( \text{number in which 1 is a part, necessarily once} \Big)
\]

Given a partition \( \pi \) in one of these sets, subtract 1 from each part to form a new partition \( \pi' \). In the first case, \( \pi' \) is a partition of \( n - d \) with \( d \) distinct parts. In the second case, \( \pi' \) is a partition of \( n - d \) with \( d - 1 \) distinct parts, since the part 1 has disappeared. Hence
\[
q_{n,d} = q_{n-d,d} + q_{n-d,d-1}.
\]
26.5.4

Show that the mysterious identity can also be written in the form

\[ \prod_{i=1}^{\infty} (1 - x^i) = \sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{1}{2} m (3m - 1)}. \]

**Solution**

For each positive integer \( m \) we have

\[ (-1)^{-m} x^{\frac{1}{2} (-m)(3(-m)-1)} = (-1)^m x^{\frac{1}{2} m (3m+1)}. \]

When \( m = 0 \) the term \((-1)^m x^{\frac{1}{2} m (3m-1)}\) is 1. Hence the sum over all integers (positive, negative and zero) as given in the question, is the same as the form stated in the text.
26.6 The calculation of $p(n)$

26.6.1

Extend the table as far as $p(20)$.

Solution

Using the same layout as in Table 26.6.1, the numbers are as follows.

<table>
<thead>
<tr>
<th></th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(n - 1)$</td>
<td>135</td>
<td>176</td>
<td>231</td>
<td>297</td>
<td>385</td>
<td>490</td>
</tr>
<tr>
<td>$p(n - 2)$</td>
<td>101</td>
<td>135</td>
<td>176</td>
<td>231</td>
<td>297</td>
<td>385</td>
</tr>
<tr>
<td>$p(n - 5)$</td>
<td>42</td>
<td>56</td>
<td>77</td>
<td>101</td>
<td>135</td>
<td>176</td>
</tr>
<tr>
<td>$p(n - 7)$</td>
<td>22</td>
<td>30</td>
<td>42</td>
<td>56</td>
<td>77</td>
<td>101</td>
</tr>
<tr>
<td>$p(n - 12)$</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>22</td>
</tr>
<tr>
<td>$p(n - 15)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

|    | 176 | 231 | 297 | 385 | 490 | 627 |
26.6.4

Show that the number of additions and subtractions required to compute $p(n)$ by the method of this section is $O(n^{3/2})$.

Solution

Let us say that the $i$th stage in the computation is the computation of $p(i)$, $1 \leq i \leq n$. According to the previous result, this requires not more than $2i^{1/2} \leq 2n^{1/2}$ additions and multiplications. Since there are $n$ stages, the total number is not more than $n \times 2n^{1/2} = O(n^{3/2})$. 