25.1 Power series and their algebraic properties

25.1.1

Find the first four terms in the power series for

(i) \( \frac{1 + 4x}{1 + 5x + x^2} \);

(ii) \( \frac{2 + 6x + x^2}{3 + x + 5x^2 + x^3} \).

(You may assume that the coefficients belong to the field \( \mathbb{R} \) of real numbers.)

**Solution** Using the method of long division, as explained in the text, you should obtain

(i) \( 1 - x + 4x^2 - 19x^3 \);

(ii) \( \frac{2}{3} + \frac{16}{9}x - \frac{37}{27}x^2 - \frac{221}{81}x^3 \).

Note that in the second example the coefficients are clearly going to be rational numbers. It may be helpful to take out a factor \( \frac{2}{3} \) at the start, so that the leading coefficients are 1.
25.1.2

Show that the inverse of $1 + x$ in $\mathbb{R}[[x]]$ is

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots,$$

where the coefficient of $x^n$ is $(-1)^n$.

**Solution** Since the answer is given, it is easy to check by the rule for multiplying power series.

$$(1 + x)(1 - x + x^2 - x^3 + \cdots) = 1 + (1 - 1)x + (1 - 1)x^2 + \cdots = 1.$$
25.2 Partial fractions

25.2.1

Use the cover-up rule to find the partial fraction decomposition of

(i) \[ \frac{3 + 4x}{(1 - x)(2 + x)}; \]

(ii) \[ \frac{2 + x + x^2}{(1 + x)(2 + x)(3 + x)}. \]

Solution

(i) \[ \frac{7}{3} \frac{1}{1 - x} + \frac{(-5/3)}{2 + x}, \]

(ii) \[ \frac{1}{1 + x} + \frac{(-4)}{2 + x} + \frac{4}{3 + x}. \]
25.2.2

Find the partial fraction decompositions of

(i) \( \frac{1 + 3x}{1 - 3x^2 + 2x^3} \),

(ii) \( \frac{-5 + 3x}{6 - 11x + 6x^2 - x^3} \).

**Solution** Using either of the methods explained in the text leads to the answers:

(i) \( \frac{1 + 3x}{1 - 3x^2 + 2x^3} = \frac{-2/9}{2x + 1} + \frac{1/9}{x - 1} + \frac{4/3}{(x - 1)^2} \),

(ii) \( \frac{-5 + 3x}{6 - 11x + 6x^2 - x^3} = \frac{-1}{1 - x} + \frac{-1}{2 - x} + \frac{2}{3 - x} \).
25.2.5

Find the partial fraction decomposition over the complex field $\mathbb{C}$ of $(1 - x^4)^{-1}$.

Solution  We have

$$\frac{1}{1 - x^4} = \frac{-1}{(x-1)(x+1)(x-i)(x+i)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-i} + \frac{D}{x+i}.$$  

The coefficients can be found from the cover-up rule:

$$A = \frac{-1}{(1+1)(1-i)(1+i)} = -\frac{1}{4}, \quad B = \frac{-1}{(-1-1)(-1-i)(-1+i)} = \frac{1}{4},$$

$$C = \frac{-1}{(i-1)(i+1)2i} = -\frac{i}{4}, \quad D = \frac{i}{4}.$$  

Hence

$$\frac{1}{1 - x^4} = \frac{1}{4} \left( \frac{1}{1 + x} + \frac{1}{1 - x} + \frac{i}{i - x} + \frac{i}{i + x} \right).$$
Solutions to Chapter 25 Exercises

in Discrete Mathematics by Norman L. Biggs;
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25.3 The binomial theorem for negative exponents

25.3.1

Write down, and simplify wherever possible, the coefficient of

(i) $x^3$ in $(1 + 2x)^7$;
(ii) $x^n$ in $(1 - x)^{-4}$;
(iii) $x^{2r}$ in $(1 - x)^{-r}$.

Solution

(i) $2^3 \times \binom{7}{3} = 280$

(ii) \[
\binom{4 + n - 1}{n} = \binom{n + 3}{n} = \binom{n + 3}{3} = \frac{1}{6} (n + 1) (n + 2) (n + 3)
\]

(iii) \[
\binom{r + 2r - 1}{2r} = \binom{3r - 1}{2r} = \binom{3r - 1}{r - 1}.
\]
25.3.2

Write down the first four terms and the general term in the power series \((1-x)^{-3}\).

Solution

Using the general formula for the coefficients,

\[
(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + \cdots
\]

\[
= 1 + 3x + 6x^2 + 10x^3 + \cdots.
\]

The general term is

\[
\binom{n+2}{n} x^n = \binom{n+2}{2} x^n = \frac{1}{2} (n+1)(n+2)x^n.
\]
Solutions to Chapter 25 Exercises

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25.4 Generating functions

25.4.1

Use the generating function method to find a formula for $u_n$ when the sequence $(u_n)$ is defined by

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+2} - 4u_{n+1} + 4u_n = 0 \quad (n \geq 0).$$

**Solution** Using the recursion, we have

$$U(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots = 1 + x + (4u_1 - 4u_0)x^2 + (4u_2 - 4u_1)x^3 + \cdots = 1 + x + 4x(U(x) - 1) - 4x^2U(x).$$

Rearranging we get the generating function

$$U(x) = \frac{1 - 3x}{1 - 4x + 4x^2}.$$

In partial fractions, this is

$$U(x) = \frac{1}{2} \left( \frac{3}{1 - 2x} - \frac{1}{(1 - 2x)^2} \right).$$

In this form the coefficients of $x^n$ are given explicitly by Theorem 25.3:

$$u_n = \frac{3}{2}2^n - \frac{1}{2}(n + 1)2^n = - (n - 2)2^{n-1}.$$
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25.4.2

Suppose \( A(x) \) is the generating function for the sequence \((a_n)\). What are the generating functions for the sequences \((p_n)\), \((q_n)\), and \((r_n)\) defined as follows?

(i) \( p_n = 5a_n \),
(ii) \( q_n = a_n + 5 \),
(iii) \( r_n = a_{n+5} \).

Solution

(i)

\[
P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \\
= 5a_0 + 5a_1 x + 5a_2 x^2 + 5a_3 x^3 + \cdots \\
= 5A(x).
\]

(ii)

\[
q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots \\
= (a_0 + 5) + (a_1 + 5)x + (a_2 + 5)x^2 + (a_3 + 5)x^3 + \cdots \\
= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + 5(1 + x + x^2 + x^3 + \cdots) \\
= A(x) + \frac{5}{1 - x}.
\]

(iii)

\[
R(x) = A(x) - \left(\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{x^5}\right).
\]
25.5 The homogeneous linear recursion

25.5.1

Use the auxiliary equation method to find a formula for \( u_n \) when the sequence \((u_n)\) is defined by

(i) \( u_0 = 1, \quad u_1 = 3, \quad u_{n+2} - 3u_{n+1} - 4u_n = 0 \quad (n \geq 0); \)

(ii) \( u_0 = 2, \quad u_1 = 0, \quad u_2 = -2, \quad u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 0 \quad (n \geq 0); \)

(iii) \( u_0 = 1, \quad u_1 = 0, \quad u_2 = 0, \quad u_{n+3} - 3u_{n+1} + 2u_n = 0 \quad (n \geq 0). \)

Solution

(i) \( \frac{1}{5} (4^{n+1} + (-1)^n) \)

(ii) \( 5 - 2^{n+2} + 3^n \)

(iii) \( \frac{1}{9} (8 - 6n + (-2)^n) \)
25.5.2

Professor McBrain climbs stairs in an erratic fashion. Sometimes he takes two stairs in one stride, sometimes only one. Find a formula for $b_n$, the number of different ways in which he can climb $n$ stairs.

**Solution**  The key step is to split the ways of climbing $n$ stairs into two parts: those in which the last stride is one stair, and those in which the last stride is two stairs. In the first part, the number of ways is $b_{n-1}$, since the first $n-1$ stairs can be climbed in any of the approved ways; in the second part the number is $b_{n-2}$. Hence

\[ b_n = b_{n-1} + b_{n-2} \quad (n \geq 3). \]

It is clear that $b_1 = 1$ and $b_2 = 2$, so we have a recursion for $b_n$.

The recursion can be solved directly, but it is simpler to notice the relationship with the Fibonacci numbers $f_n$ (p.32 and Ex. 19.2.2). In fact $b_n = f_{n+1}$, so the formula is

\[ b_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]. \]
25.6 Non-homogeneous linear recursions

25.6.1
Show that the generating function for the sequence \((u_n)\) defined by the recursion

\[
\begin{align*}
u_0 &= 1, \\
u_{n+1} - 2u_n &= 4^n \quad (n \geq 0)
\end{align*}
\]

is

\[
U(x) = \frac{1 - 3x}{(1 - 2x)(1 - 4x)}.
\]

Deduce that \(u_n = 2^{2n-1} + 2^{n-1}\).

**Solution**  Using the standard method:

\[
U(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots = u_0 + (2u_0 + 1)x + (2u_1 + 4)x^2 + (2u_2 + 4^2)x^3 + \cdots
\]

\[
= u_0 + 2xU(x) + x(1 + 4x + 4^2x^2 + \cdots).
\]

Putting \(u_0 = 1\) and rearranging:

\[
(1 - 2x)U(x) = 1 + \frac{x}{1 - 4x} = \frac{1 - 3x}{1 - 4x},
\]

\[
U(x) = \frac{1 - 3x}{(1 - 2x)(1 - 4x)}.
\]

In partial fractions:

\[
U(x) = \frac{1}{2} \left( \frac{1}{1 - 2x} + \frac{1}{1 - 4x} \right).
\]

So \(u_n = \frac{1}{2} (2^n + 4^n) = 2^{n-1} + 2^{2n-1}\).