21.1 Definitions and examples

21.1.1
Which of the following are groups of permutations of the set \{1, 2, 3, 4, 5\}, that is, which of them are subgroups of \(S_5\)?

(i) \{ (12345), (124)(35) \}.
(ii) \{ id, (12345), (13524), (14253), (15432) \}.
(iii) \{ id, (12)(34), (13)(24), (14)(23) \}.
(iv) \{ id, (12)(345), (135)(24), (15324), (12)(45), (134)(25), (143)(25) \}.

Solution

(i) Not a subgroup because the identity is not a member.
(ii) Yes.
(iii) Yes.
(iv) Not a subgroup, because the order (7) does not divide 120.
21.1.2

Find the orders of the following permutations, considered as elements of the symmetric group $S_8$:

(i) \((1235)(48)(67)\);

(ii) \((12)(35)(48)(67)\);

(iii) \((13672)(458)\).

Solution

(i) 4.

(ii) 2.

(iii) 15.
21.1.4

List all the symmetries of a regular pentagon, regarded as permutations of the corners 1, 2, 3, 4, 5, labelled in cyclic order.

Solution  There are ten symmetries:
identity, id;
rotation through 72° clockwise, (12345);
rotation through 144° clockwise, (13524);
rotation through 216° clockwise, (14253);
rotation through 288° clockwise, (15432);
reflection in axis through corner 1, (25)(34);
reflection in axis through corner 2, (13)(45);
reflection in axis through corner 3, (24)(15);
reflection in axis through corner 4, (12)(35);
reflection in axis through corner 5, (14)(23).
21.1.5

Find the group of automorphisms of the graph given by the following adjacency list. (A picture will help.)

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**Solution**  A rough sketch should indicate that the graph has triangular symmetry. There is a ‘triangle’ 123, the vertices 4, 5, 6 are adjacent to 1, 2, 3 respectively, and both vertices 7, 8 are adjacent to all of 4, 5, 6. In fact there are 12 automorphisms altogether, as listed below.

- id
- (123) (456) (78)
- (132) (465) (78)
- (23) (56) (78)
- (13) (46) (78)
- (12) (45) (78)
21.2 Orbits and stabilizers

21.2.1

Write down all the automorphisms of the graph shown in Fig. 21.1. (There are only two of them!) Show that the group of automorphisms induces a partition of the vertex set into three orbits.

Solution The automorphisms are the identity and (15)(24). The orbits are \{1, 5\}, \{2, 4\} and \{3\}. 

21.2.2

Let $G$ be the group of automorphisms of the tree shown in Fig. 21.4a, acting on the set $X$ of vertices. Determine the orbits of $G$ on $X$.

**Solution**  The symmetry of the diagram (Fig. 21.4a) makes it easy to spot the automorphisms. The vertex 3 is the only vertex of degree three, and must therefore be fixed by every automorphism. There are six automorphisms altogether:

$$
\begin{align*}
\text{id} & \\
(176)(254) & \\
(167)(245) & \\
(17)(25) & \\
(16)(24) & \\
(67)(45).
\end{align*}
$$

Two vertices are in the same orbit if and only if there is an automorphism that transforms one into the other. It follows by inspection of the list that the orbits are

$$
\begin{align*}
\{1, 6, 7\}, \{2, 4, 5\}, \{3\}.
\end{align*}
$$

Note that vertices in the same orbit must have the same degree.
21.3 The size of an orbit

Label the corners of a regular tetrahedron $T$ as 1, 2, 3, 4. Write down the permutations corresponding to the twelve rotational symmetries of $T$ and verify that the group obtained is the alternating group $A_4$.

**Solution** Some of the rotational symmetries are described in the Example at the end of Section 21.3. There are two non-identity rotations fixing each vertex; these are

$$(123), (132), (124), (142), (134), (143), (234), (243).$$

Remembering also the identity, you have nine members of the group, so three others are needed. If you can spot them geometrically, that is good. If not, you can use the fact that the composition of any two of the rotations above must also belong to the group. For example,

$$(123)(124) = (13)(24),$$

must belong to the group. In fact this represents rotation through $180^\circ$ about an axis passing through the mid-points of the edges 13 and 24. There are three similar axes, and the rotations are

$$(13)(24), (14)(23), (12)(34).$$

Since all twelve permutations are even, the group is the alternating group $A_4$ (see p.281).
21.3.3

Let $V$ be the vertex-set of the graph $\Gamma$ shown in Fig. 21.6, and let $G$ be the automorphism group of $\Gamma$. Determine the orbits of $G$ on $V$, and compute the orders of $G_a$, $G_b$, and $G$.

Solution

The orbits are

$$\{a, f\}, \{b, c, d, e\}.$$ 

$G_a = \{\text{id}, (bc), (de), (bc)(de)\}$, $|G_a| = 4$.

$G_b = \{\text{id}, (de)\}$, $|G_b| = 2$.

Checking:

$$|G| = |G_a| \times |G_a| = 4 \times 2 = 8;$$

$$|G| = |G_b| \times |G_b| = 2 \times 4 = 8.$$
Solutions to Chapter 21 Exercises

in *Discrete Mathematics* by Norman L. Biggs;
2nd Edition 2002

21.4 The number of orbits

21.4.1

Show that there are just five different necklaces which can be constructed from five white beads and three black beads. Sketch them.

**Solution**  Consider the necklace as a regular 8-gon. Without taking account of symmetry, we can put the three black beads at any three of the eight vertices, and so the number of possible configurations is

\[
\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.
\]

The group of symmetries comprises the identity, seven rotations and eight reflections. The identity fixes all 56 configurations. Any choice of positions for the three black beads leaves three gaps of total length 5. These cannot be of equal size, so no rotation can fix them.

There are two types of reflections: (i) reflection in an axis that passes through opposite vertices, and (ii) reflection in an axis that passes through the mid-points of opposite sides. A configuration fixed by type (i) must have one black bead on the axis and the two others must correspond on either side of the axis. There are two ways of placing the first black bead and then three ways of placing the other two, making 6 configurations fixed by the reflection. A configuration fixed by a reflection of type (ii) would need to have equal numbers of black beads on either side of the axis, which is impossible (since there are 3 black beads). Hence

\[
\sum_{g \in G} |F(g)| = 56 + (4 \times 6) = 80.
\]

Dividing by |G| = 16, the number of orbits is 5.
21.4.3

Let $V$ be the vertex set of the binary tree shown in Fig. 21.9, and let $G$ be the group of automorphisms of the tree. Write down the elements of $G$ (as permutations of $V$), and verify that Theorem 21.4 holds in this case.

Solution  There are 8 automorphisms:

$$
\begin{align*}
\text{id} & : (23)(46)(57) \\
(45) & : (23)(47)(56) \\
(57) & : (23)(4657) \\
(45)(67) & : (23)(4756)
\end{align*}
$$

It is clear, both from the diagram and the list of permutations, that there are three orbits:

$$
\{1\}, \{2,3\}, \{4,5,6,7\}.
$$

The number of fixed points of an automorphism is just the number of 1-cycles in the corresponding permutation, so it follows from the results given above that

$$
\frac{1}{|G|} \sum |F(g)| = \frac{1}{8} (7 + 5 + 5 + 3 + 1 + 1 + 1 + 1) = \frac{24}{8} = 3.
$$
21.5 Representation of groups by permutations

21.5.1
Let $G = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ be the group whose table is given below. Write down the permutations $\hat{g}_i$ (1 ≤ $i$ ≤ 8) as defined in Theorem 21.5 and hence determine the order of each element of $G$. (It is convenient to use the subscript $i$ instead of $g_i$ when writing out the permutations: thus $\hat{g}_2 = (1234)(5678)$.)

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Solution  The permutations are as follows.

$\hat{g}_1 = \text{id}, \quad \hat{g}_2 = (1234)(5678), \quad \hat{g}_3 = (13)(24)(57)(68),$
$\hat{g}_4 = (1432)(5876), \quad \hat{g}_5 = (1537)(2846), \quad \hat{g}_6 = (1638)(2547),$
$\hat{g}_7 = (1735)(2648), \quad \hat{g}_8 = (1836)(2745).$

All have order 4, except $\hat{g}_1$ and $\hat{g}_3$, which have orders 1 and 2 respectively.
21.5.2

Let $G$ be the group of symmetries of a regular hexagon. In each of the following cases, say whether or not the representation of $G$ by permutations of the given set $X$ is faithful.

(i) $X =$ corners;
(ii) $X =$ sides (considered as unordered pairs of corners);
(iii) $X =$ diagonals;
(iv) $X =$ perpendicular bisectors of the sides.

Solution

(i) and (ii) are faithful, because every symmetry is represented by a different permutation of the six corners, or the six sides.

(iii) is not faithful, because the identity and the rotation through 180° are both represented by the identity permutation of the three diagonals.

(iv) is not faithful, as in (iii).
21.5.3

Show that if we have an unfaithful representation of a group $G$ then there is an element $g \neq 1$ in $G$ for which $\hat{g} = id$.

**Solution**  If the representation is unfaithful, there is a pair $g_1 \neq g_2$ such that $\hat{g}_1 = \hat{g}_2$. Then letting $g = g_1^{-1}g_2$ we have

$$g \neq 1 \quad \text{and} \quad \hat{g} = id = \hat{1}.$$
Solutions to Chapter 21 Exercises

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21.6 Applications to group theory

21.6.1

How many permutations in \( S_8 \) commute with \((135)(24)(67)(8)\)?

Solution The general formula for \(|C(x)|\) is given in the text. In this case
\( x = (135)(24)(67)(8) \), with \( \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1 \), so the answer is
\[
1^1 \times 2^2 \times 3^1 \times 1! \times 2! \times 1! = 24.
\]
21.6.5
Show that the order of the centralizer of \( \pi = (123)(45) \) in \( S_7 \) is 12. Prove that the centralizer is isomorphic to \( C_6 \times C_2 \), where \( C_6 \) is generated by \( \pi \) and \( C_2 \) is generated by a permutation \( \sigma \), which is to be found.

**Solution**
By the formula

\[
|C(\pi)| = 1^2 \times 2^1 \times 3^1 \times 2! \times 1! \times 1! = 12.
\]

Clearly \( \langle \pi \rangle \) is contained in the centralizer, and \( \langle \pi \rangle \approx C_6 \). We require another permutation, of order 2, that commutes with \( \pi \). It is fairly clear that this must be \( \sigma = (67) \). Furthermore,

\[
\langle \pi \rangle \times \langle \sigma \rangle \approx C_6 \times C_2
\]

is a group of order 12, as required.