Solutions to Chapter 18 Exercises

in *Discrete Mathematics* by Norman L. Biggs;
2nd Edition 2002

18.1 Digraphs

18.1.1 In the adjacency list for a *digraph* we place *y* in column *x* whenever 
(*x*, *y*) is an arc. Sketch the digraph whose adjacency list is

\[
\begin{array}{cccccc}
  a & b & c & d & e & f \\
  d & a & b & b & f & a \\
  e & c & & & e \\
\end{array}
\]

Find a directed path from *c* to *f*, and a directed cycle starting and ending at *d*.

Solution

![Diagram of a digraph with nodes a, b, c, d, e, f and arrows indicating the directions of the arcs. The path is marked as c, b, d, e, f, and the cycle as defad.](image-url)
18.1.2 In the following table, there is a + in row $i$ and column $j$ if $i$ beats $j$, and a – if $j$ beats $i$. Find a directed path containing all the vertices of the tournament.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</thead>
<tbody>
<tr>
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</tr>
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<td>7</td>
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<td>+</td>
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</tr>
</tbody>
</table>

**Solution**

\[ 1, 2, 4, 5, 6, 7, 8, 3, 9 \]
18.2 Networks and critical paths

18.2.1 Calculate the latest times $L(v)$ and the float times $F(u, v)$ for the project described in the Example above. Find a critical path, and draw up a schedule for the project showing the alternative start times for the activities which are not critical.

Solution

$$
\begin{array}{ccccccc}
v & s & p & q & r & z & t \\
E(v) & 0 & 3 & 9 & 4 & 14 & 16 \\
L(v) & 0 & 3 & 9 & 5 & 14 & 16 \\
\end{array}
$$

Floats $(s, v)$ $(s, p)$ $(r, z)$ $(r, q)$ $(p, q)$ $(q, z)$ $(z, t)$ $(q, t)$

1 0 3 1 0 0 0 2

Critical path $s, p, q, z, t$

Critical activities $\alpha_2, \alpha_5, \alpha_6, \alpha_7$
18.3 Flows and cuts

18.3.1 Find a flow $f^*$ in Fig. 18.3 for which $\text{val}(f^*) = 10$. Why is this the maximum possible value?

Solution

\[
\begin{array}{c}
\text{Arc: } (s,a) (s,b) (s,c) (a,d) (b,d) (c,d) (a,t) (d,t) (t,c) \\
\text{Flow: } 5 \ 2 \ 3 \ 2 \ 2 \ 0 \ 3 \ 4 \ 3
\end{array}
\]

Max flow since the cut \{s,b\}, \{a,c,d,t\} has capacity 10.
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18.4 The max-flow min-cut theorem

18.4.1 The diagram (Fig. 18.5) represents a network, and the numbers on the arcs are their capacities. A flow $f$ is defined as follows.

$$(x, y): (s, a) (s, b) (s, c) (a, b) (a, d) (b, c) (b, d) (b, e) (c, e) (d, t) (e, t)$$

$$f(x, y): 5 \quad 6 \quad 0 \quad 0 \quad 5 \quad 1 \quad 2 \quad 3 \quad 1 \quad 7 \quad 4$$

(i) What is the value of $f$?
(ii) Find an $f$-augmenting path and compute the value of the augmented flow.
(iii) Find a cut with capacity 12.
(iv) What can you deduce?

![Fig. 18.5](image-url) A network, showing the capacity of each arc.

Solution

![Solution Diagram](solution-image-url)

(i) $\text{val } f = 5 + 6 + 0 = 11 \ (7 + 4 \text{ check})$
(ii) $f$-augmenting path is $s, c, b, d, t$
(iii) A cut with capacity 12 is as shown:

\{s, b, c, e\}, \{a, d, t\}

(iv) $f^*$ is max flow, val 12, cut is min cut, capacity 12.
18.5 The labelling algorithm for network flows

18.5.1 Starting from the zero flow, use the labelling algorithm (by hand!) to find the maximum flow in the network illustrated in Fig. 18.10.

![Network diagram](image)

**Fig. 18.10** Find the maximum flow.

**Solution**

![Labelling steps](image)

Tree (V) does not reach t. Max flow = 55, min cut is (s, 3), where s = all vertices reached in (V), that is all except t.