17.1 Relations and bipartite graphs
17.1.1 Let $X = \{2, 3, 5, 7, 11\}$, $Y = \{99, 100, 101, 102, 103\}$, and define $x$ and $y$ to be related whenever $x$ is a divisor of $y$. Draw the bipartite graph representing this relation, and verify that Theorem 17.1 is satisfied.

Solution

![Bipartite Graph]

Degrees in $X$: $2 + 2 + 1 + 0 + 1 = 6$.

Degrees in $Y$: $2 + 2 + 0 + 2 + 0 = 6$.

Check: number of edges $= 6$. 

146
Solutions to Chapter 17 Exercises

in Discrete Mathematics by Norman L. Biggs;
2nd Edition 2002

17.1.2 The complete bipartite graph $K_{r,s}$ is the bipartite graph $(X \cup Y, E)$ in which $|X| = r$, $|Y| = s$, and every pair $xy$ with $x \in X$ and $y \in Y$ is an edge.

(i) What is the degree of each vertex in $X$?
(ii) What is the degree of each vertex in $Y$?
(iii) How many edges are there in $K_{r,s}$?
(iv) Describe in ordinary language the relation which $K_{r,s}$ represents.
(v) Show that for any $s \geq 1$ the graph $K_{1,s}$ is a tree.
(vi) Show that $K_{r,s}$ is not a tree whenever $r \geq s \geq 2$

Solution

(i) $s$.
(ii) $r$.
(iii) $rs$.
(iv) Every $x$ is related to every $y$.
(v) $K_{1,s}$ is connected: because every vertex is joined to $x_1$.
$K_{1,s}$ has no cycles: because any cycle would need two $X$-vertices.
(vi) If $r \geq s \geq 2$ there is a cycle $x_1 y_1 x_2 y_2 x_1$. 

147
17.1.3 Is the graph illustrated in Fig. 17.2 bipartite?

Solution No, because there is an odd cycle (see Theorem 15.7.2).
Solutions to Chapter 17 Exercises

in *Discrete Mathematics* by Norman L. Biggs;
2nd Edition 2002

17.2 Edge colourings of graphs

17.2.1 What is the least number of colours required for an edge colouring of

(i) the complete graph $K_4$;
(ii) the complete graph $K_5$;
(iii) the cube graph (Fig. 15.12)?

**Solution**

(i)

![Triangle diagram with 3 colours labeled as $\alpha$, $\beta$, $\gamma$]

(ii)

![Dodecahedron diagram with 5 colours labeled as $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$]

(4 would mean $\frac{5}{4}$ of each colour)

(iii)

![Cube diagram with 3 colours labeled as $\alpha$, $\beta$, $\gamma$]

149
Solutions to Chapter 17 Exercises

in  *Discrete Mathematics*  by  Norman L. Biggs;
2nd Edition 2002

17.2.3  Prove that for any positive integer $n$ the complete bipartite graph $K_{n,n}$ has an edge colouring with $n$ colours.

**Solution**  Give $x_i, y_j$ colour $i - j \pmod n$. 
17.2.4 Show that the graph illustrated in Fig. 17.4 is bipartite and construct an edge colouring of it using only three colours.

**Fig. 17.4** A bipartite graph.

**Solution**
Solutions to Chapter 17 Exercises
in *Discrete Mathematics* by Norman L. Biggs;
2nd Edition 2002

17.3 Application of edge colouring to latin squares
17.3.1 Use the edge-colouring method to extend the following latin rectangle
to a $5 \times 5$ latin square.

$$
\begin{align*}
A & B & C & D & E \\
C & D & B & E & A \\
B & C & E & A & D
\end{align*}
$$

Solution

$$
\begin{align*}
A & B & C & D & E \\
C & D & B & E & A \\
B & C & E & A & D
\end{align*}
$$

Final square is

$$
\begin{align*}
A & B & C & D & E \\
C & D & B & E & A \\
B & C & E & A & D \\
D & E & A & C & B \\
E & A & D & B & C
\end{align*}
$$
17.3.2 Find all values of $Q$ for which the rectangle $R_1$ can be extended to a $6 \times 6$ latin square. Show that $R_2$ cannot be so extended, whatever value $Q$ has.

\[
\begin{array}{c c c c}
R_1: & A & B & C & D \\
& F & E & A & B \\
C & D & F & A \\
D & A & B & Q \\
\end{array}
\]

\[
\begin{array}{c c c c}
R_2: & A & B & C & D \\
& F & E & A & B \\
& B & D & F & A \\
D & A & B & Q \\
\end{array}
\]

**Solution** Condition for extendibility is $N_R(s) \geq 4 + 4 - 6 = 2$.

For $R_1$, discounting $Q$, we have

\[
s: \quad A \quad B \quad C \quad D \quad E \quad F \\
N_{R_1}(s): \quad 4 \quad 3 \quad 2 \quad 3 \quad 1 \quad 2 \\
\]

\[\therefore \text{Must have} \quad Q = E \text{ for extendibility.}\]

For $R_2$, discounting $Q$, we have

\[
s: \quad A \quad B \quad C \quad D \quad E \quad F \\
N_{R_2}(s): \quad 4 \quad 4 \quad 1 \quad 3 \quad 1 \quad 2 \\
\]

\[\therefore \text{Need an extra} \quad C \text{ and an extra} \quad E.\]
17.4 Matchings

17.4.1 Use Hall’s condition to show that the graph in Fig. 17.9 has no complete matching.

Solution Please note that the answer given in the back of the book is WRONG. It will be corrected as soon as possible.

We require a subset $A \subseteq X$ such that $|J(A)| < |A|$. In fact,

$$J \{x_1, x_2, x_3, x_4\} = \{y_2, y_4, y_5\},$$

so $\{x_1, x_2, x_3, x_4\}$ is such a set.
17.4.2 Let $M$ be the matching denoted by heavy lines in Fig. 17.9.

(i) Find an alternating path for $M$ beginning at $x_2$.
(ii) Use it to construct a matching $M'$ with $|M'| = 4$.
(iii) Check that there is no alternating path for $M'$.
(iv) Is $M'$ a maximum matching?

Solution

(i)

(ii) We remove $x_5y_5$ from $M$, and add $x_2y_5$ and $x_5y_1$, so that

$$M' = \{ x_2 y_5, x_3 y_2, x_4 y_4, x_5 y_1 \}.$$ 

(iii) Alternating path must begin at $x_1$ since $x_1$ is the only vertex unmatched. The possibilities are as follows.

(iv) Yes. (One vertex must be left over, because of Exercise 1.)
17.5 Maximum matchings

17.5.1 Let $G = (X \cup Y, E)$ be the bipartite graph with $X = \{a, b, c, d, e\}$, $Y = \{v, w, x, y, z\}$, and $E = \{av, ax, bw, bz, cw, cy, cz, dy, dz, ez\}$. Use the algorithmic method to find a complete matching in $G$, starting from the matching $M = \{av, bz, cy\}$.

Solution

Tree (from $d$) is

$w$ is unmatched, $\therefore$ switch on $dyew$.
New matching is $M' = \{av, bz, cw, dy\}$.
Tree (from $e$) is

$x$ is unmatched, \therefore switch on $ezbvax$.  
New matching is \{ax, bv, cw, dy, ez\}.
17.6 Transversals for families of finite sets

17.6.1 Let \( S \) be the family of sets \{a, b, l, e\}, \{t, e, s, t\}, \{s, t, a, b\}, \{s, a, l, e\}, \{t, a, l, e\}, \{s, a, l, t\}. Find a transversal for \( S \).

Solution

<table>
<thead>
<tr>
<th>rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>able</td>
</tr>
<tr>
<td>test</td>
</tr>
<tr>
<td>stab</td>
</tr>
<tr>
<td>sale</td>
</tr>
<tr>
<td>tale</td>
</tr>
<tr>
<td>salt</td>
</tr>
<tr>
<td>a</td>
</tr>
<tr>
<td>l</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>e</td>
</tr>
<tr>
<td>t</td>
</tr>
<tr>
<td>s</td>
</tr>
</tbody>
</table>

\( \therefore \) Let \( a, l, b, e, t, s \) represent the sets in the given order.

165
Solutions to Chapter 17 Exercises

in Discrete Mathematics by Norman L. Biggs;
2nd Edition 2002

17.6.3 Prove that the family of sets \{a, m\}, \{a, r, e\}, \{m, a, r, e\}, \{m, a, s, t, e, r\}, \{m, e\}, \{r, a, m\} has no transversal, by showing explicitly that Hall’s condition does not hold.

Solution There are five sets \textit{am}, \textit{are}, \textit{mare}, \textit{me}, \textit{ram} containing only the four elements \textit{a, e, m, r}.