Solutions to Chapter 15 Exercises

in Discrete Mathematics by Norman L. Biggs;
2nd Edition 2002

15.1 Graphs and their representation

15.1.1

Three houses A, B, C each has to be connected to the gas, water, and electricity supplies: G, W, E. Write down the adjacency list for the graph which represents this problem, and construct a pictorial representation of it. Can you find a picture in which the lines representing the edges do not cross?

Solution The adjacency list is:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>G</th>
<th>W</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>G</td>
<td>G</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>W</td>
<td>W</td>
<td>W</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>E</td>
<td>E</td>
<td>E</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

A picture, with lots of crossings, is as follows.

```
A -- G
|
B -- W
|
C -- E
```

You will find that, by redrawing, it is possible to remove some of the crossings, but it is impossible to remove all of them. This is just an 'experimental fact' at this stage: fortunately it can be proved.
15.1.2

The pathways in a formal garden are to be laid out in the form of a wheel graph $W_n$, whose vertex set is $V = \{0, 1, 2, \ldots, n\}$ and whose edges are

$$
\{0, 1\}, \{0, 2\}, \ldots, \{0, n\}, \\
\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}.
$$

Describe a route around the pathways which starts and ends at vertex 0 and visits every vertex once only.

**Solution**  It is helpful to draw a diagram depicting the graph as a ‘wheel’, with the vertex 0 at the center and the other vertices in the order 1, 2, 3, $\ldots$, $n$ around the rim. Then it is clear that a route of the required form is obtained by visiting the vertices in the order

$$0, 1, 2, \ldots, n, 0.$$
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15.1.3
For each positive integer \( n \) we define the complete graph \( K_n \) to be the graph with \( n \) vertices in which each pair of vertices is adjacent. How many edges has \( K_n \)? For which values of \( n \) can you find a pictorial representation of \( K_n \) with the property that the lines representing the edges do not cross?

**Solution** The number of edges is just the number of unordered pairs of vertices, that is

\[
|E(K_n)| = \binom{n}{2} = \frac{1}{2} n(n - 1).
\]

The graphs \( K_1, K_2, K_3, K_4 \) can be drawn without crossings fairly easily, although you will have to be careful with the last edge in \( K_4 \).

However, you will not be able to find a drawing of \( K_5 \) without crossings. [See the comment on Ex. 15.1.1].
15.2 Isomorphism of graphs

15.2.1

Prove that the graphs shown in Fig. 15.5 are not isomorphic.

Solution The first graph has two 3-cycles (see Ex 15.1.4 for the meaning of this statement). If it were isomorphic with the second graph, that graph would have 3-cycles also, but in fact there are none.
15.2.2

Find an isomorphism between the graphs defined by the following adjacency lists. (Both lists specify versions of a famous graph, known as Petersen’s graph. See also Ex. 15.8.3.)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<tr>
<td>b</td>
<td>1</td>
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<td>4</td>
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<td>0</td>
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<td>f</td>
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<td>g</td>
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<td>i</td>
<td>2</td>
<td>6</td>
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<td>8</td>
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<td>9</td>
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<tr>
<td>j</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Solution In this case there are many possible isomorphisms.

One way to construct one is to start with an arbitrary assignment, say $\alpha(a) = 4$. Then it must be the case that the vertices adjacent to $a$ correspond to the vertices adjacent to 4 in some order; try $\alpha(b) = 3$, $\alpha(e) = 6$, $\alpha(f) = 5$. Continuing in this way, and checking the adjacency condition at each stage, it turns out that the following is an isomorphism.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>
15.3 Degree

Is it possible that the following lists are the degrees of all the vertices of a graph? If so, give a pictorial representation of such a graph. (Remember that there is at most one edge joining each pair of vertices.)

(i) 2, 2, 2, 3.
(ii) 1, 2, 2, 3, 4.
(iii) 2, 2, 4, 4, 4.
(iv) 1, 2, 3, 4.

Solution

(i) No. There is only one odd value, whereas any graph must have an even number of vertices with odd degree.

(ii) Yes. If we name the vertices a,b,c,d,e in the given order of degrees, the following adjacency list has the required property.

```
     a b c d e
     e d d b a
     e e c b
     e c
     d
```

(iii) No. There are five vertices altogether; three of them have degree 4, which means that each of them must be adjacent to every other vertex. Hence every vertex must have degree three, at least.

(iv) No. Here there are only four vertices altogether, so none of them can have degree 4.
15.3.2

If $G = (V, E)$ is a graph, the complement $\bar{G}$ of $G$ is the graph whose vertex set is $V$ and whose edges join those pairs of vertices which are not joined in $G$. If $G$ has $n$ vertices and their degrees are $d_1, d_2, \ldots, d_n$, what are the degrees of the vertices of $\bar{G}$?

Solution

\[ n - 1 - d_1, \; n - 1 - d_2, \; \ldots, \; n - 1 - d_n. \]
15.4 Paths and cycles

15.4.1

Find the number of components of the graph whose adjacency list is

\[
\begin{array}{cccccccc}
  a & b & c & d & e & f & g & h & i & j \\
  f & c & b & h & c & a & b & d & a & a \\
i & g & e & g & i & c & f & f \\
j & g & j & e
\end{array}
\]

**Solution** In this example, a sketch of the graph will reveal that there are three components, with vertex-sets \( \{a, f, i\} \), \( \{b, c, g\} \), \( \{d, h\} \). A general algorithm will be described in Section 16.4.
15.4.2

How many components are there in the graph of April’s party (Section 15.1)?

Solution  The vertex of degree 8 is joined to all the other vertices except the vertex O of degree 0. So there are two components, \{O\} and the rest.
15.4.3

Find a Hamiltonian cycle in the graph formed by the vertices and edges of an ordinary cube.

Solution  Although the general problem of finding a Hamiltonian cycle is hard, in this case it is easy to find one by trial and error. For example, using the diagram on p.395, a Hamiltonian cycle is

1, 2, 3, 7, 6, 5, 8, 4, 1.
15.5 Trees

15.5.1
There are six different (that is, mutually non-isomorphic) trees with six vertices: draw them.

Solution It is helpful to have a systematic approach. For example, you can try to make a list in order, according to the largest degree of a vertex, say $K$.

Since the there are six vertices, $K$ is at most 5. When $K = 5$ the vertex of degree 5 is adjacent to the remaining five vertices, and we get the first tree illustrated below. When $K = 4$, the vertex of degree 4 is adjacent to four others; the sixth vertex must be adjacent to one of these four, and clearly all possibilities are the same (isomorphic). When $K = 3$ there are three mutually non-isomorphic possibilities, and when $K = 2$ only one.

The complete list is shown below.
15.6 Colouring the vertices of a graph

15.6.1
Find the chromatic numbers of the following graphs:

(i) a complete graph $K_n$;
(ii) a cycle graph $C_{2r}$ with an even number of vertices;
(iii) a cycle graph $C_{2r+1}$ with an odd number of vertices.

Solution

(i) Since each vertex of $K_n$ is adjacent to every other vertex, all the vertices must have different colours. Hence $\chi(K_n) = n$.

(ii) Suppose we try with just two colours. Give the vertices in cyclic order alternate colours: 1, 2, 1, 2, …. The only problem might occur when colouring the last vertex, which is adjacent to the first one: but if the number of vertices is even, then the last vertex gets colour 2 and the first vertex colour 1, so we have a proper colouring with two colours.

(iii) Here the method adopted in the previous part fails, because the first and last vertices both get colour 1. We need a third colour for the last vertex, and so $\chi(C_{2r+1}) = 3$. 
15.6.2 Determine the chromatic numbers of the graphs depicted in Fig. 15.11.

Solution It is worth remembering that in order to show that \( \chi(G) = k \), there are two steps: (i) find a colouring with \( k \) colours; (ii) show that there is no colouring with fewer than \( k \) colours.

For the first graph it is easy to find a colouring with 3 colours. To show that 2 colours are not enough, consider the 5-cycle formed by a 'diagonal' and four edges joining its ends. (See also Theorem 15.7.2.)

For the second graph, we need 4 colours. To show that 3 is not enough, note that 3 must be used on the 'outer' pentagon, and this forces a fourth colour in the centre.

For the third graph, a colouring with 4 colours is shown below. There is a set of 4 vertices that are mutually adjacent (forming a \( K_4 \) subgraph) so fewer than 4 colours is impossible.
15.7 The greedy algorithm for vertex colouring

15.7.1
Find orderings of the vertices of the cube graph (Fig. 15.12) for which the greedy algorithm requires 2, 3, and 4 colours, respectively.

Solution You will need to begin by labelling the vertices of the cube graph in some arbitrary fashion. For definiteness, we shall use the labeling as in Fig. 27.8 (p.395).

Colouring the vertices greedily in the order

\[ 1 3 2 4 5 7 6 8 \]

assigns the colours \[ 1 2 2 2 2 1 1. \]

Similarly, the order

\[ 1 7 2 3 4 5 6 8 \]

assigns the colours \[ 1 2 3 2 2 3 3; \]

and the order

\[ 1 7 2 3 8 5 6 4 \]

assigns the colours \[ 1 2 3 2 3 4 4. \]

In fact, any ordering will result in the use of not more than 4 colours (see Theorem 15.7.1).