10.1 Principles of counting

10.1.1 The rules for the University of Forlonia five-a-side soccer competition specify that the members of each team must have birthdays in the same month. How many mathematics students are needed in order to guarantee that they can raise a team?

Discussion  This is an example of the generalised pigeonhole principle mentioned in the text.

Solution  The ‘boxes’ in this application are the months of the calendar, so in the notation of the generalised pigeonhole principle we have $n = 12$ (months in the year).

We want to find the number $m$ of students such that there are at least five allocated to one ‘box’, so (in the same notation) we want $r + 1 = 5$, i.e. $r = 4$.

Thus we want the least natural number $m$ such that $m > 12 \times 4$, that is $m = 49$. The least number of students needed is therefore 49.

Remark  When people use mathematics to answer real-world questions, the process has several stages: translate the question into a mathematical problem; solve the mathematical problem; translate the solution back into a statement about - hopefully an answer to - the original question. In doing mathematics exercises, the step which corresponds to the last stage is to check the original question, and make sure that one has answered it: hence the last sentence.
10.1.2 What is wrong with the following argument? Since one-half of the numbers \( n \) in the range \( 1 \leq n \leq 60 \) are multiples of 2, 30 of them cannot be primes. Since one-third of the numbers are multiples of 3, 20 of them cannot be primes. Hence at most 10 of them are primes.

Discussion There are several things wrong with the argument, all of which come under the heading colloquially known as ‘double-counting’. When counting is being used in mathematics (as in life), it is important to take care that it is used in an appropriate fashion.

Solution The main error is that some of the 30 multiples of 2 are also amongst the 20 multiples of 3, and hence were counted twice (‘double-counted’) in the given argument: 6 is such a number. In mathematical terms, we would say that the sets \( A \) of multiples of 2 (in the given range) and \( B \) of multiples of 3 were not disjoint, that is \( A \cap B \neq \emptyset \); accordingly, we cannot apply Theorem 10.1 to conclude that we have found \( 30 + 20 = 50 \) numbers in the range which cannot be prime.

The argument also involves two other (small) instances of double-counting: 2 is one of the 30 multiples of 2, but is itself prime, so ought not to be excluded from the list of possible primes, and on similar grounds neither should 3 be excluded.
10.1.4 Show that in any set of 10 people there are either four mutual friends or three mutual strangers.

Discussion This is similar to the Example in Section 10.1, and the same approach can be used. In fact, the result of that Example plays an important role. When you have worked out what is happening here, you may wish to make a more general conjecture. (This area of mathematics is known as Ramsey Theory.)

Solution If α is any one of the ten people, partition the remaining nine into two sets: F, the friends of α, and N the not-friends of α. Then either |F| ≥ 6 or |N| ≥ 4, since otherwise the two sets would contain at most 5 + 3 = 8 people.

Case 1 If |F| ≥ 6 choose any six members of F. Applying the result given in the Example above, this set contains either 3 mutual friends or 3 mutual strangers. If there are 3 mutual friends, these three, together with α, make a set of 4 mutual friends, as required; otherwise there are three mutual strangers.

Case 2 If |N| ≥ 4, choose any four members of N. If these four are mutual friends the result is proved. If they are not, there is at least one pair, say β, γ who are strangers, and these two, together with α, make a set of 3 mutual strangers.
10.2 Counting sets of pairs

10.2.1 In Dr Cynthia Angst’s Calculus class, 32 of the students are boys. Each boy know five of the girls in the class and each girl knows eight of the boys. How many girls are in the class?

Discussion This is an application of Theorem 10.2.(ii), as in the Example in the section. The argument of course depends on the fact (or at least the assumption) that if girl A knows boy Z then boy Z knows girl A, and vice versa.

Solution 1 (Informal, i.e. without explicit use of Theorem 10.2; hence lengthy.) In an (imaginary) rectangular array or table with the boys listed as columns and the girls as rows, and with a tick whenever a pair know each other, we would find $32 \times 5 = 160$ ‘boy knows girl’ ticks. If the number of girls is $n$ then we would also find $8 \times n$ ‘girl knows boy’ ticks, and since the numbers must be equal, we find that $n = 20$. There are twenty girls in the class.

Solution 2 If $G$ is the set of girls then, following Theorem 10.2.(ii):

$$5 \times 32 = 8 \times |G|, \text{ so } |G| = 20.$$ 

Hence there are twenty girls in the class.
10.2.2 Suppose we have a number of different subsets of \( \mathbb{N}_8 \), with the property that each one has four members, and each member of \( \mathbb{N}_8 \) belongs to exactly three of the subsets. How many subsets are there? Write down a collection of subsets which satisfies the conditions.

Discussion As in Ex. 10.2.1, this question is most efficiently answered by applying Theorem 10.2(ii). It is important, for clarity of thought and of communication, to define the symbols which are used if there is any room for uncertainty about them.

It is probably a useful, if cautionary, experience, to attempt the second part of the question by trial and error, in order to grasp how hard it can be (even though 8 is much smaller than other numbers for which similar tasks may be required), unless we know a workable general method of approach. The problem is one which involves Designs, which is the topic of Sections 11.6 and 11.7. In the solution here we ‘borrow’ some ideas from the discussions there.

Solution Let \( X \) be the set of relevant subsets of \( \mathbb{N}_8 \).

By Theorem 10.2(ii):

\[
4 \times |X| = 3 \times |\mathbb{N}_8| = 24, \text{ so } |X| = 6.
\]

Hence there six subsets in the collection \( X \).

One example of a ‘real-life’ situation in which this problem might arise can be found at the start of Section 11.6, as can the solution:

\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{1, 3, 5, 7\}, \{2, 4, 6, 8\}, \{1, 2, 4, 7\}, \{3, 5, 6, 8\}.  

10.2.3 Is it possible to find a collection of subsets of $\mathbb{N}_8$ such that each one has three members and each member of $\mathbb{N}_8$ belongs to exactly five of the subsets?

**Discussion** This exercise illustrates that results such as Theorem 10.2 can be used to test YES/NO questions, as to whether something is possible at all, as well as to calculate relevant values when the answer is YES.

**Solution** If it is possible, and if $X$ denotes the collection of relevant subsets of $\mathbb{N}_8$, then $|X|$ must be a whole number solution to the equation:

$$3 \times |X| = 5 \times 8 = 40,$$

but no such solution exists because 3 does not divide 40.
10.3 Euler’s function

10.3.1 Find the values of \( \phi(19) \), \( \phi(20) \), \( \phi(21) \).

Discussion As one might expect from its definition, calculating values of Euler’s \( \phi \)-function utilises the unique prime factorization of integers \( n > 1 \) (Section 8.6).

Solution Since 19 is prime, \( \phi(19) = 19 - 1 = 18 \), as in the text. On the other hand, we have the factorization \( 20 = 2^2 \times 5 \), and hence the integers \( x \) such that \( 1 \leq x \leq 20 \) and \( \gcd(x, 20) = 1 \) are going to be the ones in that range which are divisible neither by 2 nor by 5: the list is therefore \( 1, 3, 7, 9, 11, 13, 17, 19 \), so that \( \phi(20) = 8 \).

Similarly, \( 21 = 3 \times 7 \), and making a similar list reveals that \( \phi(21) = 12 \).
10.3.2 Show that if \( x \) and \( n \) are coprime, then so are \( n - x \) and \( n \). Deduce that \( \phi(n) \) is even for all \( n \geq 3 \).

Discussion Recall that (in a different notation) Theorem 8.4 and Ex. 8.4.2 show that natural numbers (positive integers) \( a, b \) satisfy

\[
gcd(a, b) = 1 \text{ if and only if there exist integers } u, v \text{ with } au + bv = 1.
\]

It is not very difficult to make the (trivial) modifications to the definitions, and to extend the proofs, so as to show that this result remains true if \( a, b \) are allowed to be any integers, rather than being required to be positive.

The second part of the question involves both counting arguments and ‘case-by-case’ arguments: examples of each type crop up frequently throughout mathematics.

Solution From the Discussion, if \( x \) and \( n \) are coprime there exist integers \( u, v \) such that \( ux + vn = 1 \), and then \((-u) \times (n-x) + (u + v) \times n = 1\), so (by the Discussion again) \( n-x \) and \( n \) are also coprime.

If \( n \geq 3 \) and \( n \) is even, suppose that \( n = 2m \). We can match up 1 with \( n-1 \) to get a pair of (distinct) suitable integers that are coprime to \( n \); if \( \phi(n) > 2 \) let \( x \) be the next smallest natural number such that \( \gcd(x,n) = 1 \), and pair it up with \( n-x \), then repeat the procedure as long as necessary. It is easy to check that each pair contains one integer \( y \) with \( 1 \leq y < m \) and one integer \( n-y \) with \( m < n-y < n \); clearly neither 2 nor \( m \) is coprime to \( n \), so our pairs cover all possibilities, and therefore \( \phi(n) \) is even.

If \( n \geq 3 \) and \( n \) is odd, suppose that \( n = 2m+1 \). In similar fashion, we divide up the relevant integers in pairs \( x \) and \( n-x \) of distinct numbers each coprime to \( n \), but this time within the ranges \( 1 \leq x \leq m \) and \( m < n-x \leq 2m \) respectively; again we conclude that \( \phi(n) \) is even.
10.4 Functions, words and selections

10.4.1 How many national flags can be constructed from three equal vertical strips, using the colours red, white, blue and green? (It is assumed that colours can be repeated, and that one vertical edge of the flag is distinguished as the ‘flagpole side’.)

Discussion The underlying ideas here are exactly the same as for Ex. 10.2.5.

Solution Ordering the stripes, say in order away from the flagpole side, there are three successive choices of colour to be made. For each choice there are four possibilities, and no choice affects the options available for any other, so there are $4^3 = 64$ possible national flags.
10.4.2 Write down all the subsets of the set \{a, b, c, d\} and use the correspondence given in the Example to check that your list is complete.

**Solution** We can list the sets, and the corresponding numbers (with columns running a, b, c, d from left to right), as follows:

<table>
<thead>
<tr>
<th>Subset</th>
<th>Numbers</th>
<th>Corresponding Subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>\emptyset</td>
<td>0000</td>
<td>{a, b, c, d}</td>
</tr>
<tr>
<td>{a}</td>
<td>1000</td>
<td>{d}</td>
</tr>
<tr>
<td>{b}</td>
<td>0100</td>
<td>{c}</td>
</tr>
<tr>
<td>{a, b}</td>
<td>1100</td>
<td>{c, d}</td>
</tr>
<tr>
<td>{a, c}</td>
<td>1010</td>
<td>{b, d}</td>
</tr>
<tr>
<td>{a, d}</td>
<td>1001</td>
<td>{b, c}</td>
</tr>
<tr>
<td>{a, b, c}</td>
<td>1110</td>
<td>{b, c, d}</td>
</tr>
<tr>
<td>{a, b, d}</td>
<td>1101</td>
<td>{a, c, d}</td>
</tr>
</tbody>
</table>
10.5 Injections as ordered selections without repetition

10.5.1 In how many ways can we select a batting order of 11 from a pool of 14 cricketers?

Discussion When selecting finitely many things from a finite set, we need to know whether we want them in order (as we do here) and whether or not repetition is allowed (it is not here, by contrast with Section 10.4). If, as in this case, we want an ordered listing, without repetitions, then Theorem 10.5 is the result to use.

Solution In the notation of Theorem 10.5, we have \( m = 11 \) and \( n = 14 \), so the number of possible selections is

\[
\begin{align*}
  n \times (n - 1) \times (n - 2) \times \ldots \times (n - m + 1) &= 14 \times 13 \times 12 \times \ldots \times 4,
\end{align*}
\]

and a calculator shows that this is equal to 14,529,715,200.

Remark Depending on the available functions, working this out accurately can possibly be tricky. When problems arise, it is often worth devising a minor modification of the calculation which is feasible. In this case, a ‘scientific notation’ calculation gave only an approximate result, but by eliminating the factors 10 and 5, and replacing the 4 at the right hand end by a 2, we obtain \( 1/100 \) times the correct result:

\[
14 \times 13 \times 12 \times 11 \times 9 \times 8 \times 7 \times 6 \times 2 = 145,297,152 = 14,529,715,200/100.
\]
10.5.2 How many four-letter words can be made from an alphabet of 10 symbols if there are no restrictions on spelling except that no letter can be used more than once?

Solution A word is four characters in order, and no repetitions are permitted, but there are no other restrictions. In the notation of Theorem 10.5 we have \( m = 4 \) and \( n = 10 \), so the requisite number is

\[
n \times (n - 1) \times (n - 2) \times \ldots \times (n - m + 1) = 10 \times 9 \times 8 \times 7 = 5040.
\]

Hence there are 5040 possible words.
10.5.3 Explain briefly how you would make a systematic list of all the ordered selections, without repetition, of three things from the set \{a, b, c, d, e\}.

Discussion In a question which asks for a brief explanation of how to do something, it is worth thinking carefully just how ‘brief’ a comprehensive explanation could be. More often than not, it is not necessary actually to compile the list (which in this case would have $5 \times 4 \times 3 = 60$ items, a number which - though not impossible - is certainly very unwieldy).

Solution One systematic approach would be to take advantage of the point, emphasised in the text, that we have five choices for the first selection, then four for the second, then three for the third (and that’s it).

Thus we could list all possibilities in which the first entry was $a$, sub-ordered so that all those with $b$ as second entry were at the beginning (giving $abc$, $abd$ and $abe$), then moving on through those in which the second entry was $c$ until we reached those beginning $ae$. Then we could repeat the process by listing all possibilities in which the first entry was $b$, and continuing through those beginning with $c$ and with $d$ until we had exhausted all the possibilities beginning with $e$: then we stop.

Remark There are lots of trivial variations on the method just described. One non-trivial observation about them is that methods for the 5-element set \{a, b, c, d, e\} apply (repeatedly) similar methods for 4-element sets such as \{b, c, d, e\} and \{a, b, c, d\}, which depend on similar (though here merely trivial) methods for 3-element sets such as \{c, d, e\}, which in a different question might depend on similar methods for 2-element sets . . . . Patterns of the type just mentioned are very common when dealing with topics like these.
10.6 Permutations

10.6.1 Write down the cycle notation for the permutation which effects the rearrangement

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ 3 \quad 5 \quad 7 \quad 8 \quad 4 \quad 6 \quad 1 \quad 2 \quad 9 \]

Discussion There are several important points and ‘tricks’ that are useful when doing calculations with permutations of finite sets:

- When converting a permutation to cycle form or when ‘multiplying’ (composing) two or more permutations, the best approach (as explained in the text) is:

  pick the smallest number involved (or the left hand symbol, or any other sensible selection criterion which can be repeatedly applied);

  track what it ‘moves’ to, then track what that moves to, and so on, until you get back to the number you started with;

  write out the resulting cycle;

  repeat if there are any numbers left, until they have all been dealt with.

- As mentioned in the text, a cycle of the type (3), which ‘leaves 3 where it was’, can be omitted from the description (in cycle terms) of a permutation, but it is much better practice not to do so when first meeting this material.
• Be aware that powers of a cycle need not have the same pattern of cycle-lengths in the cycle decompositions of their own:

for example, \((12)^2 = (1)(2)\) and equals \(id\) if we are working in \(S_2\);

for example, \((123)^3 = (1)(2)(3)\) and equals \(id\) if we are working in \(S_3\);

for example, \((1234)^2 = (13)(24) \neq id\) whatever we are working in.

• Both for cycles in their own right and for cycles as factors of more complicated permutations, if you want to find a power of the cycle or permutation, just ‘count along’ the cycle in question:

for example, \((12345)^1\) takes 1 to 2, 2 to 3, and so on, so \((12345)^2\) does the same rearrangement twice, \(i.e.\) takes 1 to 3, 2 to 4, 3 to 5, 4 to 1 and 5 to 2, and (provided that you remember to ‘go round the back’ at the right hand end, \(e.g.\) from 4 to 5 to 1 or from 5 to 1 to 2) you can work out the \(s\)th power by counting along \(s\) times.

• Exactly the same method works for the \(s\)th power of a product of cycles, \(provided that\) the different cycles involved have no numbers (or symbols) in common.

• If the ‘non-overlapping’ condition just mentioned is not met, the ‘counting along’ method for powers of products of cycles does not work, nor does it (in most cases) for products of distinct permutations, even if they are written in cycle form. Ex. 2 contains examples of this situation.

• If \(\sigma\) is a cycle with \(r\) numbers then so is \(\sigma^{-1}\), and it can be described as the a cycle of the same numbers in the opposite order;

for example, \((1234)^{-1}\) can be written as \((4321)\), although some mathematicians may opt for the form \((1432)\) on the (fairly irrelevant) grounds that in it 1 comes first.

• It is sometimes useful, as in Ex. 2, to compute products by ‘tracking’ elements using a diagram with arrows.

These comments apply to all of the questions in Exercises 10.6.

**Solution**  Tracking 1, we see that 1 goes to 3 which goes to 7 which
goes (back) to 1, so one cycle will be (137). Next, track 2: we get $2 \rightarrow 5 \rightarrow 4 \rightarrow 8 \rightarrow 2$, and hence another factor will be (2548). That leaves 6 and 9, which don’t ‘move’; hence

$$\sigma = (137)(2548)(6)(9).$$
10.6.2 Let σ and τ be the permutations of \{1, 2, \ldots, 8\} whose representations in cycle notation are
\[ \sigma = (123)(456)(78), \quad \tau = (1357)(26)(4)(8). \]
Write down the cycle notations for \(\sigma \tau, \tau \sigma, \sigma^2, \sigma^{-1}, \tau^{-1}\).

**Solution** Using arrows to compute \(\sigma \tau\) (and remembering that it means ‘τ first, then σ’), we get:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\tau & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 6 & 5 & 4 & 7 & 2 & 1 & 8 \\
\sigma & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 4 & 6 & 5 & 8 & 3 & 2 & 7 \\
\end{array}
\]

and then we have to track how this composite moves the various numbers. Clearly 1 is not moved, so look at \(2 \to 4 \to 5 \to 8 \to 7 \to 2\); hence we get the cycles (1) and (24587). Next, look at \(3 \to 6 \to 3\), and the final cycle turns out to be (36). Hence
\[
\sigma \tau = (1)(24587)(36).
\]

Using a similar method to calculate \(\tau \sigma\) (‘σ first’) we get:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\sigma & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 3 & 1 & 5 & 6 & 4 & 8 & 7 \\
\tau & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
6 & 5 & 3 & 7 & 2 & 4 & 8 & 1 \\
\end{array}
\]

Tracking, we get (in abridged notation) \(1 \to 6 \to 4 \to 7 \to 8 \to 1\), and hence (16478). Next we get \(2 \to 5 \to 2\), and hence (25). The final cycle is clearly (3), so
\[ \tau \sigma = (16478)(25)(3). \]

We have \((123)^2 = (132)\), \((456)^2 = (465)\) and \((78)^2 = (7)(8)\), so that (by the Discussion at Ex. 1.)
\[ \sigma^2 = (132)(645)(7)(8). \]

Also by the Discussion at Ex. 1:
\[ \sigma^{-1} = (321)(654)(87) = (132)(465)(78). \]

and
\[ \tau^{-1} = (7531)(62)(4)(8) = (1753)(26)(4)(8). \]

Remark. Unlike the case of familiar numbers, for permutations the order in which we multiply two of them together usually does make a difference: in this case, \(\sigma \tau\) is completely different from \(\tau \sigma\).
Solutions to Chapter 10 Exercises
in Discrete Mathematics by Norman L. Biggs;
2nd Edition 2002

10.6.3 Solve the problem posed in the Example when there are 20 cards arranged in five rows of four.

Discussion The method of tackling this problem is exactly like that in the Solution to the Example, but there is a much larger problem of notation: would (for instance) the symbol sequence \((19)\) stand for the \((2-)\)cycle which swaps 1 and 9 or for the \((1-)\)cycle which fixes the number 19? Our notation uses \(x_r\) for \(1 \leq r \leq 20\) to represent the relevant positive integers, so that \((x_1x_9)\) is the swapping \((2-)\)cycle just mentioned, while \((x_{19})\) is the cycle which fixes \(x_{19}\) (or simply fixes 19).

The above method is only one of many ways to tackle this notational difficulty.

Solution With the notation just described, but otherwise proceeding just as in the text, the operation being considered takes the numbers (symbols) as arranged on the left of the following table to the arrangement on its right.

\[
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  x_5 & x_6 & x_7 & x_8 \\
  x_9 & x_{10} & x_{11} & x_{12} \\
  x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{17} & x_{18} & x_{19} & x_{20}
\end{array}
\quad \begin{array}{cccc}
  x_1 & x_6 & x_{11} & x_{16} \\
  x_2 & x_7 & x_{12} & x_{17} \\
  x_3 & x_8 & x_{13} & x_{18} \\
  x_4 & x_9 & x_{14} & x_{19} \\
  x_5 & x_{10} & x_{16} & x_{20}
\end{array}
\]

As in the text, let \(\pi\) be the permutation defined by \(\pi(i) = j\) if card \(j\) turns up in the former position of card \(i\). Again as in the text, the top left and bottom right corners give the degenerate cycles \((x_1)\) and \((x_{20})\).

A little rough working shows that we also get two further cycles:

\[(x_2x_6x_7x_{12}x_{18}x_{19}x_{17}x_5)\text{ and } (x_{3}x_{11}x_{13}x_{4}x_{16}x_{19}x_{15}x_{14}x_{9}).\]

Accordingly,

\[\pi = (x_1)(x_2x_6x_7x_{12}x_{18}x_{19}x_{17}x_5)(x_{3}x_{11}x_{13}x_{4}x_{16}x_{19}x_{15}x_{14}x_{9})(x_{20}).\]
As in the text, there are two degenerate cycles and two others of identical lengths (here 9), so we have to repeat the permutation nine times: $\pi^9 = id$, while of course $\pi^n \neq id$ for (positive) values of $n$ smaller than 9.