6.1 Counting as a bijection

6.1.1 In each of the following cases write down a formula for a bijection \( f : \{1, 2, \ldots, m\} \rightarrow X \), for an appropriate value of \( m \).

(a) \( X = \{2, 4, 6, 8, 10\} \).
(b) \( X = \{3, 8, 13, 18, 23, 28\} \).
(c) \( X = \{10, 17, 26, 37, 50, 65, 82, 101\} \).
(d) \( X = \{k \in \mathbb{N} \mid \text{the } k\text{th day of this month is a Monday}\} \).

Discussion If the elements of \( X \) are numbers, they should be counted, and if necessary listed in increasing order (already done here for (a), (b) and (c)). To find a formula, the next thing to do is to look at the successive differences between successive set members, e.g. 2 in (a), 5 in (b), and some more complicated pattern in (c); if \( X \) has two or more elements then there will typically be several possible formulae, any one of which gives a suitable bijection. But it is usually best to choose an example, often unique, with one of the properties:

if \( j, k \in X \) then \( j < k \) implies \( f(j) < f(k) \), or
if \( j, k \in X \) then \( j < k \) implies \( f(j) > f(k) \).

In each part \( m \) is the size of the set \( X \), so is fixed, and in writing out a formula we must choose a different symbol for the variable. The answer to (d) will, of course, depend on the date that the exercise is done.

Solution (a) There are five elements of \( X \), so \( m = 5 \), and \( f(k) = 2k \) for all \( k \in \{1, 2, 3, 4, 5\} \) is a suitable formula.
(b) There are six elements of $X$, so $m = 6$, and $g_1(k) = 5k - 2$ for all $k \in \{1, 2, 3, 4, 5, 6\}$ is a suitable formula, as also is $g_2(k) = 33 - 5k$ for all $k \in \{1, 2, 3, 4, 5, 6\}$.

(c) There are eight elements of $X$, so $m = 8$, and the sequence of differences is $17 - 10 = 7$, then $26 - 17 = 9$, $37 - 26 = 11$, and so on; it is easy to see that the differences themselves are increasing by two each time. It can be shown that, in these circumstances, the function must have degree two, and a suitable formula is:

$$h(k) = (k + 2)^2 + 1 \text{ for all } k \in \{1, 2, 3, 4, 5, 6, 7, 8\}.$$  

(d) In August 2002 (CE) we find $m = 4$, and a suitable formula is $j(k) = 7k - 2$ for all $k \in \{1, 2, 3, 4\}$. 
6.2 The size of a set

6.2.1 Give an example of a function from \( X = \{1, 2, 3\} \) to \( Y = \{0, 1\} \). Let \( F \) denote the set of all functions \( f : X \rightarrow Y \). Show that \( |F| = 8 \), by listing the members of \( F \), calling them \( f_1, f_2, \ldots, f_8 \).

Discussion In preparing such a list, it is helpful to consider the pattern of changes of the different possible values of \( f_j(1), f_j(2) \) and \( f_j(3) \) as \( j \) runs through the values from 1 to 8. This approach is adopted in the following table; you may find it helpful to look at the pattern in the three right-hand columns.

Solution We can list eight possible functions by listing the values that they take, in \( Y \), for the three possible values of \( x \in X \), as follows:

\[
\begin{array}{c|ccc}
  x & 1 & 2 & 3 \\
  \hline
  f_1 & 0 & 0 & 0 \\
  f_2 & 0 & 0 & 1 \\
  f_3 & 0 & 1 & 0 \\
  f_4 & 0 & 1 & 1 \\
  f_5 & 1 & 0 & 0 \\
  f_6 & 1 & 0 & 1 \\
  f_7 & 1 & 1 & 0 \\
  f_8 & 1 & 1 & 1 \\
\end{array}
\]

It is clear that, for distinct subscripts \( i, j \) with \( 1 \leq i, j \leq 8 \) there exists at least one \( x \in X \) such that \( f_i(x) \neq f_j(x) \), so all eight functions are distinct, and hence \( |F| \geq 8 \).

On the other hand, for any function \( f : X \rightarrow Y \) the values \( f(1), f(2) \) and \( f(3) \) must be shown on some line of this table, be-
cause there are no other possible choices (see Ex. 2 below). This means that $|F| \leq 8$ and, combining the two inequalities, it follows that $|F| = 8$. 
6.2.2 If $|X| = m$ and $|Y| = 2$, what is the size of the set $F$ of all functions $f : X \to Y$? [The answer itself is fairly easy. When you have spotted it, write out a formal proof using the principle of induction.]

Discussion The case $m = 3$ was discussed in Ex. 1, where it was shown that $|F| = 8 = 2^3$.

If the elements of $X$ are listed, e.g. if $X = \{x_1, x_2, \ldots, x_m\}$, then there are two choices for $f(x_1)$, two choices for $f(x_2)$, two choices for $\ldots$, and finally two choices for $f(x_m)$. Moreover, the choices are independent, in the sense that where $i, j$ are distinct subscripts the choice for $x_i$ does not affect, and is not affected by, the choice for $x_j$. This means that the total number of choices ought to be $2 \times 2 \times \ldots \times 2$ ($m$ factors), and hence the best guess is that $|F| = 2^m$.

It is convenient to adopt the conventions regarding notation that are explained at the start of the proof.

**Theorem** If $X, Y$ are sets, if $|X| = m \in \mathbb{N}$ and $|Y| = 2$, and if $F = \{f : X \to Y \mid f \text{ is a function}\}$ then $|F| = 2^m$.

**Proof** We use induction on $m$, and can assume that $Y = \{0, 1\}$, that

$$X_m = \{x_1, x_2, \ldots, x_m\}$$

(where $X_m$ denotes the version of $X$ that has $m$ elements), and that $F_m = \{f : X_m \to Y \mid f \text{ is a function}\}$.

Induction basis If $m = 1$ then there are two functions $f : X_1 \to Y$, namely $f_1(x_1) = 0$ and $f_2(x_1) = 1$, so the theorem is true when $m = 1$.

Induction hypothesis Suppose that $k \in \mathbb{N}$ and that $|F_k| = 2^k$.

Induction step With the chosen notation, it is natural to regard
$X_k$ as a subset of $X_{k+1}$: let $j : X_k \to X_{k+1}$ be the inclusion map (Section 5.2), i.e. $j(x_r) = x_r \in X_{k+1}$ for $1 \leq r \leq k$. If $g, h \in F_{k+1}$, then the composites $gj, hj \in F_k$, and it is clear that $gj = hj$ if and only if the only element, if any, of $X_{k+1}$ for which $g$ and $h$ take different values is $x_{k+1}$. (This process is often called restriction of $g, h$ to $X_k$.) Since there are two possible values of $g(x_{k+1})$ (and of $h(x_{k+1})$), the elements of $F_{k+1}$ come in pairs, with each element of a pair having the same restriction to $X_k$, but elements from different pairs having different restrictions. Hence

$$|F_k| \geq |F_{k+1}|/2. \quad (A)$$

On the other hand, if $f \in F_k$ then we can use $f$ to define two ‘new’ elements of $F_{k+1}$, namely

- $f_0$, defined by $f_0(x_{k+1}) = 0$ and $f_0(x_r) = f(x_r)$ for $1 \leq r \leq k$;
- $f_1$, defined by $f_1(x_{k+1}) = 1$ and $f_1(x_r) = f(x_r)$ for $1 \leq r \leq k$.

(This process is often known as defining extensions of $f$.)

Clearly $f_0j = f_1j = f$, i.e. the restrictions of $f_0$ and of $f_1$ are each the original $f$. It follows that each $f \in F_k$ gives rise to two distinct elements of $F_{k+1}$, and hence

$$|F_k| \leq |F_{k+1}|/2. \quad (B)$$

Combining the inequalities (A) and (B), $|F_k| = |F_{k+1}|/2$. By the inductive hypothesis, $|F_k| = 2^k$, so $|F_{k+1}| = 2^{k+1}$.

By the principle of induction, $|F_m| = 2^m$ for all $m \in \mathbb{N}$.

**Remark** The restriction/extension method used here works for any finite set $Y$ with two or more elements. If, for instance, $Y$ has $n \geq 2$ elements then (with the same notation) each $f \in X_k$ has exactly $n$ distinct extensions to $X_{k+1}$, and is the restriction to $X_k$ of exactly $n$ distinct elements of $F_{k+1}$. It follows easily that, in these circumstances, there are $n^m$ distinct functions $f : X \to Y$. 

8
6.3 A counting problem

6.3.1 A very basic ‘counting principle’, which we shall discuss in more detail in Chapter 10, concerns the relationship between

\[ |A|, |B|, |A \cup B|, \text{ and } |A \cap B|. \]

By looking at simple examples, try to formulate this relationship in general terms.

Discussion  The pictorial representation of sets, and their unions and intersections, by means of Venn diagrams was discussed in Section 2.3. If we think of the area of the shape representing each set as (proportional to) the number of its elements, we can develop an approach to the problem based upon Venn diagrams.

Alternatively, we can (and do) give examples involving small sets, in order to study certain cases and to get a ‘feel’ for what is happening.

Solution  Case 1: let \( A = \{1, 2, 3\} \), and \( B = \{2, 3, 4\} \). Then \( A \cup B = \{1, 2, 3, 4\} \), \( A \cap B = \{1, 2\} \), and \( |A| = |B| = 3 \), \( |A \cap B| = 2 \) and \( |A \cup B| = 4 \). It is easy to check that

\[ 3 + 3 = |A| + |B| = |A \cup B| + |A \cap B| = 2 + 4 = 6. \]

Case 2: let \( A = \{a, b, c, d, e\} \), and \( B = \{b, d, f, g\} \). Then \( A \cup B = \{a, b, c, d, e, f, g\} \), \( A \cap B = \{b, d\} \), and \( |A| = 5 \), \( |B| = 4 \), \( |A \cap B| = 2 \) and \( |A \cup B| = 7 \). It is easy to check that

\[ 5 + 4 = |A| + |B| = |A \cup B| + |A \cap B| = 7 + 2 = 9. \]

These cases suggest that the general formula may be:

\[ |A| + |B| = |A \cup B| + |A \cap B|. \]
The Venn diagram for two sets looks like:

![Venn Diagram](image)

and the total shaded area (whatever type of shading) is proportional to \(|A \cup B|\), which breaks down into the areas shaded \(\quad\quad\quad\) (elements of \(B\) that do not belong to \(A\)) and \(\quad\quad\quad\quad\) (elements of \(A\) that do not belong to \(B\)) and \(\quad\quad\quad\quad\quad\) (elements of \(A \cap B\)).

If we just add the area representing \(A\) to the area representing \(B\) then we count the area shaded \(\quad\quad\quad\quad\), which represents \(A \cap B\), as part of both sets; \(i.e.,\) we count it \(\textit{twice}\). That is:

\[
\text{area representing } A + \text{area representing } B - \text{area representing } A \cap B = \text{area representing } A \cup B.
\]

If areas are proportional to sizes, this suggests the same formulae:

\[
|A| + |B| - |A \cap B| = |A \cup B|, \quad \text{i.e.} \quad |A| + |B| = |A \cap B| + |A \cup B|.
\]

\textbf{Remark} The two formulae just given are really just different ways of writing the same (correct) result. Now no two of the three sets \(A \setminus (A \cap B), B \setminus (B \cap A) = B \setminus (A \cap B)\) and \(A \cap B\) have any elements in common (in standard terminology, they are 'pairwise disjoint'); given this fact, it is easy to prove that this formula is the right one using Theorem 10.1 supplemented by Ex. 10.1.3.
6.4 Some applications of the pigeonhole principle

6.4.1 A blindfolded man has a heap of 10 grey socks and 10 brown socks in a drawer. How many must he select in order to guarantee that, among them, there is a matching pair?

Discussion The trick with questions like this is to think the right way about what is happening, a skill which is mostly acquired through practice. The following solution contains both an informal argument and a direct appeal to the pigeonhole principle.

Solution Either the first two socks that are drawn are a matching pair, when no further socks are needed, or the first two socks don’t match, in which case the third sock must match one or the other of the first pair. So the man must select three socks to be certain of getting a matching pair.

Using the pigeonhole principle, suppose that the man has two notional ‘boxes’, one for grey and one for brown socks: if three socks are drawn then one such box must contain two socks of the relevant type, while drawing fewer does not ensure that that happens.

Remark Evidently, so long as there are at least three socks in the drawer, the precise numbers available don’t really matter.
6.4.2 Let $S$ be any set of 12 natural numbers. Show that $S$ must contain two distinct numbers $s_1, s_2$ such that $s_1 - s_2$ is a multiple of 11.

Discussion The following solution applies the pigeonhole principle to notional boxes labelled ‘remainders on dividing by 11’.

Solution On dividing the numbers in $S$ by 11, we obtain 12 expressions for remainders $r_i$ in the range $0 \leq r \leq 10$: for example, $s_1 = 11 \times q_1 + r_1$. By the pigeonhole principle, since only 11 possible values are available, two different subscripts $i, j$ must give the same remainder. Changing the numbering if necessary, we can assume that $i = 1$ and $j = 2$, so that $s_1 = 11 \times q_1 + r_1$ and $s_2 = 11 \times q_2 + r_2$ with $r_1 = r_2$. Hence $s_1 - s_2 = 11 \times (q_1 - q_2)$, so is a multiple of 11.

Remark If $s_1 < s_2$ then the division by 11 would take place in the set $\mathbb{Z}$ of all integers, not in $\mathbb{N}$. This could be avoided by changing the numbering to exclude that possibility. But, though changing numbering to simplify the presentation of an argument is an important, and often used technique, it does not seem worthwhile here.
6.4.3 Let $X$ be a subset of $\{1, 2, \ldots, 2n\}$, and let $Y$ be the set of odd numbers $\{1, 3, \ldots, 2n - 1\}$. Define a function $f : X \to Y$ by the rule

$$f(x) = \text{the greatest member of } Y \text{ that exactly divides } x.$$ 

Show that if $|X| \geq n + 1$ then $f$ is not an injection, and deduce that in this case $X$ contains distinct numbers $x_1$ and $x_2$ such that $x_1$ is a multiple of $x_2$.

Solution Since $|Y| = n$, if $|X| > n$ (i.e. $|X| \geq n + 1$) then by the pigeonhole principle there must be two distinct elements of $X$ for which $f$ takes the same value; that is, there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$; let the common value be $y \in Y$, an odd natural number.

Since $x_1, x_2$ are distinct we can suppose (changing subscripts if necessary) that $x_1 > x_2$, and since $y$ is the largest odd divisor of each we must have $x_1 = 2^{m_1} \times y$ and $x_2 = 2^{m_2} \times y$ for appropriate $m_1, m_2 \in \mathbb{N}$. Since $x_1 > x_2$ we must have $m_1 > m_2$, and it then follows that $x_2$ divides $x_1$. 

6.5 Infinite sets

6.5.1 By constructing a bijection from \( \mathbb{N} \) to \( S \), show that each of the following sets \( S \) is infinite.

(i) \( S = \{ n \in \mathbb{N} \mid n \geq 10^6 \} \)

(ii) \( S = \{ n \in \mathbb{N} \mid n \text{ is a multiple of } 3 \} \)

Discussion There are infinitely many bijections in both cases, but it is usually best to look for a bijection \( f \) with the property that, if \( m, n \in \mathbb{N} \) and \( m < n \) then \( f(m) < f(n) \). It is often also important (as in (i)) to make sure that the ‘starting point’ - typically, the value \( f(1) \) - is correct.

Solution

(i) A suitable function is \( f(n) = 10^6 + (n - 1) \). [Note the starting point!]

(ii) A suitable function is \( g(n) = 3n \).
6.5.2 Prove that if $X$ is a subset of $Y$, and $X$ is infinite, then $Y$ is infinite.

Discussion At the start of the section, a set $S$ was defined to be infinite if it was not finite, and was defined as finite if either it was empty or there was a bijection $f : \mathbb{N}_m \to S$ for some $m \in \mathbb{N}$ (where the notation $\mathbb{N}_m$ means the subset $\{1, 2, \ldots, m\}$ of $\mathbb{N}$); in these circumstances, $S$ was (later) said to have size $m$. The proof below uses the ‘strong’ principle of induction (Section 4.6).

Theorem If $Y$ is a finite set, either empty or of size $m \in \mathbb{N}$ then every subset $X$ of $Y$ is either empty or of size $r$, where $r \in \mathbb{N}$ and $1 \leq r \leq m$.

Discussion This theorem is the contrapositive of the required result, hence logically equivalent to it.

Proof If $Y = \emptyset$ there is nothing to prove, since then $Y$ is its own unique subset. We fix the following notation for later in the proof:

if $f : \mathbb{N}_m \to Y$ is a bijection (so that $Y$ has size $m$), and if $X$ is a subset of $Y$, let $X' = \{m \in \mathbb{N}_m \mid f(m) \in X\}$.

With this convention the function $f' : X' \to X$ defined by $f'(x) = f(x)$ for all $x \in X'$ is easily seen to be a bijection.

Induction basis Let $f : \mathbb{N}_1 \to Y$ be a bijection and $X$ be a non-empty subset of $Y$. The set $\mathbb{N}_1 = \{1\}$ has exactly two subsets, namely $\emptyset$ and $\mathbb{N}_1$ itself, so $X' = \mathbb{N}_1$, and since $f'$ is a bijection $X' \to X$ the set $X$ is also finite, of size 1. Thus if $Y$ has size 1 then every non-empty subset of $Y$ also has size 1, i.e. the theorem is true in the case $m = 1$.

Induction hypothesis $k \in \mathbb{N}$ and the theorem is true whenever
$s \in \mathbb{N}, 1 \leq s \leq k$ and $Y$ is a finite set of size $s$.

**Induction step** Suppose $Y$ is a set of size $k + 1$ and $X$ is a non-empty subset of $Y$. There is a bijection $f : \mathbb{N}_{k+1} \to Y$, and either $f(k+1) \in X$ or (using the fact that $\mathbb{N}_k$ is a subset of $\mathbb{N}_{k+1}$) $X$ is a subset of $\{f(1), f(2), \ldots, f(k)\}$.

If this latter set is called $Y'$, and if $j$ is the standard (injective) inclusion function $j : \mathbb{N}_k \to \mathbb{N}_{k+1}$, then the composite map $f j : \mathbb{N}_k \to Y'$ is easily seen to be bijective, so that $Y'$ is finite of size $k$. Because $X$ is a subset of $Y'$, it follows by the inductive hypothesis that $X$ is finite, of size $s$ with $1 \leq s \leq k$.

In the other case, $X$ is not a subset of $Y'$, so $f(k+1) \in X$. Let $Z = \{x \in X \mid x \neq f(k+1)\}$, so that $Z$ is a subset of $Y'$, and therefore finite of size $t$, with $1 \leq t \leq k$ by the previous argument. That means there is a bijection $g : \mathbb{N}_t \to Z$, so we can define a new function $h : \mathbb{N}_{k+1} \to X$ by $h(x) = g(x)$ if $x \in \mathbb{N}_t$ and $h(t + 1) = f(k+1)$. It is easy to check that $h$ is a bijection [DETAILS?], and of course $t + 1 \leq k + 1$ since $t \leq k$, so $X$ is a finite set of size at most $k + 1$.

The theorem then follows by the (strong) principle of induction.
6.5.3 Let $X$ be a non-empty subset of $\mathbb{N}$ which has no greatest member. Show that we can choose members $x_1, x_2, \ldots$ of $X$ such that $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. Deduce that $X$ is infinite.

Discussion We shall treat this fairly formally in the ‘Theorem, Proof’ style. In addition to what was asked for in the question, we show that the elements $x_1, x_2, \ldots$ of $X$ can be chosen so that every $x \in X$ crops up as one of the $x_n$’s.

Theorem If $X$ is a non-empty subset of $\mathbb{N}$ which has no greatest member there exists a bijection $\mathbb{N} \to X$, written $n \to x_n$.

Proof Since $X$ is non-empty, it has a least element, which we can label $x_1$. Since $X$ has no greatest element, the subset $X_1 = \{ x \in X \mid x_1 < x \}$ is non-empty, so it has a least element $x_2$, and certainly $x_2 > x_1$; clearly also there is no other element $y$ of $X$ such that $x_1 < y < x_2$.

So suppose that $k \in \mathbb{N}$ and that we have found elements $x_1, x_2, \ldots, x_{k+1}$ of $X$

with the properties that $x_r < x_{r+1}$ for $1 \leq r \leq k$ and that there is no element $y$ of $X$ such that $x_1 < y < x_{k+1}$ except $x_2, \ldots, x_k$. (We have already shown that this can be done when $k = 1$.) Since $X$ has no greatest element, the subset $X_{k+1} = \{ x \in X \mid x_{k+1} < x \}$ is non-empty, so it has a least element $x_{k+2}$, and certainly $x_{k+2} > x_{k+1}$; in this case too it is clear that there is no element $z$ of $X$ such that $x_1 < z < x_{k+2}$ except for $x_2, \ldots, x_{k+1}$.

By the principle of induction, we can define an injection $\mathbb{N} \to X$, written $n \to x_n$, with the property that $x_{n+1} > x_n$ for every $n \in \mathbb{N}$. It follows that $X$ has an infinite subset, namely $Y = \{ x_n \mid n \in \mathbb{N} \}$, and then by Ex. 2 that $X$ is infinite.

(This completes the original question, but it is useful to show that $n \to x_n$ is also a surjection, and hence a bijection.) It remains
to show that every $x \in X$ crops up as one of the $x_n$'s. Suppose $x \in X$; since $X$ is a subset of $\mathbb{N}$, the set $\{y \in X \mid y \leq x\}$ is a non-empty subset of $\mathbb{N}_x$, so has at most $x$ elements. It follows (by an easy induction on $x$) that $x = x_r$ for some $r \in \mathbb{N}_x$.

**Remark** Although we chose $x_1, x_2, \ldots$ in such a way that every $x \in X$ turned up in the list, there are plenty of other sequences with similar properties except that there are ‘gaps’, i.e. some $x \in X$ are unlisted. For example, if we chose the list $y_1, y_2, \ldots$ defined by $y_j = x_{2j}$ for each $j \in \mathbb{N}$ it would clearly still be true that $y_{n+1} > y_n$ for each $n \in \mathbb{N}$, but none of the odd-numbered $x_i$’s would be on that list.
6.6 Strange properties of infinite sets

6.6.1 Show that the set $H$ consisting of all natural numbers that are multiples of 100 is countable, by defining a bijective correspondence with $\mathbb{N}$.

Discussion It is worth thinking through how one would prove, in detail, that the function in the solution is a bijection.

Solution One suitable bijection is $f(n) = 100n$ for all $n \in \mathbb{N}$. 
6.6.2 Show that the set of natural numbers whose digits (in the usual base 10 representation) are all different is finite.

Discussion We offer three solutions, illustrating different approaches to this problem: reducing it to a previously solved problem (which is often useful when doing work for assessment); a direct proof via the pigeonhole principle (yielding an upper limit to the size of the set), and via a counting technique (which gives a precise figure for how many numbers there are of this type).

Solution 1 Assuming standard properties of the base 10 representation, the set $X$ of such numbers has a largest element, namely $9,876,543,210$; by Ex. 6.5.2 $X$ is a finite set, and the proof of that result shows that $|X|$ is less than or equal to $9,876,543,210$.

Solution 2 Let $X$ be the set of such natural numbers. There are ten possible digits, namely $0, 1, \ldots, 9$. If $m \in \mathbb{N}$ needs $r \geq 11$ digits for its base 10 representation then, allocating each position ('units', 'tens', \ldots, millions, \ldots) to the digit which occupies the position, the pigeonhole principle says that two positions must be allocated to the same digit, i.e. there is a repetition.

Hence a member of $X$ can only have at most 10 digits, and for each possible position of a digit there are at most 11 choices (counting the possibility, ‘no entry’). There are some restrictions on possible combinations which represent a number in the base 10 system, e.g. a ‘no entry’ in the middle of the representation of a number is not allowed, but if we ignore the restrictions there can still only be $11^{10}$ possible numbers, so $X$ is a subset of $\mathbb{N}_{11^{10}}$, a finite set. By Ex. 6.5.2, a subset of a finite set is finite; hence $X$ is finite.

Solution 3 If $X$ is the set of such numbers, and $r \in \mathbb{N}$, let $X_r$ denote the subset - possibly empty - of elements of $X$ which require
precisely $r$ digits in the base 10 representation.

It is easy to see that $|X_1| = 9$.

Each element of $X_2$ can be written in the form $10x + y$ where $x$ is a member of $X_1$ and $y$ is a digit that is permitted given the value of $x$, and hence

$$|X_2| = |X_1| \times 9 = 9 \times 9 = 81.$$ 

Each element of $X_3$ can be written in the form $10x + y$ where $x$ is a member of $X_2$ and $y$ is a digit that is permitted given the representation of $x$, and hence

$$|X_3| = |X_2| \times 8 = 81 \times 8 = 648.$$ 

Using similar arguments, it can be shown that:

$$|X_4| = |X_3| \times 7 = 648 \times 7 = 4536;$$
$$|X_5| = |X_4| \times 6 = 4536 \times 6 = 27216;$$
$$|X_6| = |X_5| \times 5 = 27216 \times 5 = 136080;$$
$$|X_7| = |X_6| \times 4 = 136080 \times 4 = 544320;$$
$$|X_8| = |X_7| \times 3 = 544320 \times 3 = 1632960;$$
$$|X_9| = |X_8| \times 2 = 1632960 \times 2 = 3265920;$$
$$|X_{10}| = |X_9| \times 1 = 3265920 \times 1 = 3265920.$$ 

(Note that $|X_9| = |X_{10}| = 9 \times (9!)$.) If we assume it is ‘obvious’, or show, along lines used in Solution 2, that $|X_r|$ is empty for $r > 10$ then we have

$$|X| = |X_1| + |X_2| + \ldots + |X_{10}| =$$
$$9 + 81 + 648 + 4536 + 27216 + 136080 + 544320 + 1632960 + 2 \times 1632960,$$

i.e. $|X| = 13,877,690$.

Remark: Which approach was ‘best’ would depend on what it was acceptable to assume, and on what it was intended to do next. If a precise value was needed then the third solution would be appropriate (despite the lack of a proper proof that $|X_r|$ is empty for $r > 10$); if not, the second solution is (probably) less work, and the first is certainly less work (but assumes a lot about the base 10 representation that one might want to prove at some stage in the study of $\mathbb{N}$). The first solution appears to give $987,654,321$ as an upper limit on the size of $X$; for comparison, $11^{10} = 25,937,424,601$, which is
therefore *far* too high, but even the upper limit of 987,654,321 pro-
vided by the first solution does not seem very good once we know
the exact value is (merely?) 13,877,690.
6.6.3 Show that the union of any two disjoint countable sets is countable.

Discussion  According to Section 6.6 a set \( A \) is countable (short for countably infinite) if there is a bijection \( \mathbb{N} \to A \). In some mathematical usage, the term ‘countable’ is taken to mean ‘finite or countably infinite’, and it is important to be clear in which sense the term is being used in any given context.

Solution  Suppose that the sets are \( A \) and \( B \), and that the functions \( a : \mathbb{N} \to A \) and \( b : \mathbb{N} \to B \) are bijections, and that - in the familiar style - their values are written as \( a_1, a_2, \ldots \) and as \( b_1, b_2, \ldots \).

Define a function \( z : \mathbb{N} \to A \cup B \) by \( z_{2n-1} = a_n \) and \( z_{2n} = b_n \). Since \( A \cap B = \emptyset \) there is no ambiguity about this definition, and since every natural number \( m \) is either odd or even, the value of \( z \) is defined for every \( n \in \mathbb{N} \). If \( i,j \in \mathbb{N} \) and \( z_i = z_j \) then either \( i \) and \( j \) are both odd or they are both even, because \( A \cap B = \emptyset \). If they are both odd then \( a_{(i+1)/2} = a_{(j+1)/2} \) (from the definition of \( Z \)), so \( (i + 1)/2 = (j + 1)/2 \) and therefore \( i = j \). If \( i \) and \( j \) are both even then a similar argument again shows that \( i = j \). Hence \( z \) is an injection. If \( k \in \mathbb{N} \) then \( a_k = z_{2k-1} \) and \( b_k = z_{2k} \), and since \( a \) and \( b \) are surjections it follows that \( z \) is surjection.

Hence \( z \) is a bijection, so \( A \cup B \) is countably infinite, i.e. countable.