5.1 The concept of a function

5.1.1 Let $U$ denote the set of citizens of the state of Utopia. Which of the following statements correctly specify a function from $U$ to $U$? (Any assumptions you make about the Utopian civilization should be stated explicitly.)

(i) $f(x)$ is the mother of $x$.
(ii) $g(x)$ is the daughter of $x$.
(iii) $h(x)$ is the wife of $x$.

Discussion Any function $f : X \rightarrow Y$ has four important features:

• the ‘starting’ set $X$ of possible values of the ‘variable’ $x \in X$;
• the ‘destination’ set $Y$ of possible values of the function $f(x) \in Y$;
• the fact that the value $f(x)$ is specified for every $x \in X$;
• the fact that $f(x)$ is uniquely determined by $x$ and whatever rule or formula is used to define $f$.

Solution (i) If we neglect very modern innovations, such as egg donations and surrogate mothers, then every person has, or had, a unique mother, who may or may not still be alive at the time in question. Assuming that ‘citizens’ includes past citizens (both female and male!), perhaps now dead or elsewhere, then $f(x)$ is defined, uniquely, for every citizen $x$ of Utopia. We can also assume (for this question) that the mother of any Utopian is (or was) herself a Utopian, so that $f(x)$ takes values in the set of Utopian citizens. This means that $f$ has all four features just discussed, so $f$ is a function.

(ii) A citizen $x$ of Utopia might have no daughters, in which case
there would not be *any* \( g(x) \), or might have several daughters, in which case \( g(x) \) would not be *unique*. Both cases would surely arise, so \( g(x) \) fails to be a function on either count.

(iii) On the surely plausible assumption that women in Utopia do not themselves have wives, the rule for \( h(x) \) does not describe its value when \( x \) is female, so is not a function. (There might also be unmarried men, and another possible problem is that, while Utopia might be one of the societies in which only adult men were ‘citizens’, in many of those societies men also have more than one wife simultaneously, so that \( h(x) \) would then fail to satisfy the fourth criterion.)
5.1.2 Write down the values \( s(1) \), \( s(2) \), \( s(3) \), \( s(4) \), \( s(5) \), \( s(6) \) of the function (sequence) defined by the rules

\[
s(1) = 1, \quad s(2) = 2, \quad s(n + 1) = 2s(n) - s(n - 1) \quad (n \geq 2).
\]

Make a conjecture about a formula for \( s(n) \) and try to prove it by using the principle of induction.

Discussion Since the formula for \( s(n) \) \((n \geq 2)\) involves two previous values, it is likely that the ‘strong’ principle of induction will be involved in an induction proof.

Solution We are given that \( s(1) = 1 \) and \( s(2) = 2 \), so \( s(3) = 2 \times 2 - 1 = 3 \); then \( s(4) = 2 \times 3 - 2 = 4 \); so \( s(5) = 2 \times 4 - 3 = 5 \); then \( s(6) = 2 \times 5 - 4 = 6 \).

The obvious conjecture (‘guess’) is that \( s(n) = n \) for all \( n \in \mathbb{N} \), and (induction basis) that is correct for \( n = 1 \). There are several possible appropriate induction hypotheses, of which the following induction hypothesis most closely fits the pattern of the ‘strong’ principle of induction (Section 4.6):

\[
k \in \mathbb{N} \text{ and if } i \in \mathbb{N} \text{ and } 1 \leq i \leq k \text{ then } s(i) = i.
\]

The induction step is then: \( s(k + 1) = 2 \times s(k) - s(k - 1) \) (by definition), so \( s(k + 1) = 2k - (k - 1) \) (by the induction hypothesis), that is \( s(k + 1) = k + 1 \).

By the (strong) principle of induction, \( s(n) = n \) for all \( n \in \mathbb{N} \).

Remark The ‘induction step’ just given does not use the full force of the strong principle of induction. For example, at the value \( k = 5 \) it relies on the (assumed) True status of \( s(4) = 4 \) and of \( s(5) = 5 \), but does not refer explicitly to values of \( s(1), s(2) \) or \( s(3) \).
5.2 Surjections, injections, bijections

5.2.1 Suppose that the sets $A$ and $B$ are $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$, and the functions $p : A \to B$, $q : B \to B$, and $r : B \to A$ are defined as follows:

\[
\begin{align*}
 p(a) &= 3 & q(1) &= 3 & r(1) &= a \\
 p(b) &= 4 & q(2) &= 5 & r(2) &= c \\
 p(c) &= 2 & q(3) &= 1 & r(3) &= b \\
 p(d) &= 4 & q(4) &= 4 & r(4) &= a \\
 & & q(5) &= 2 & r(5) &= d
\end{align*}
\]

Which of these functions are surjections, which are injections, and which are bijections?

Discussion Many mathematicians and mathematics texts also use the phrases: ‘is surjective’ to mean ‘is a surjection’; ‘is injective’ to mean ‘is an injection’; and ‘is bijective’ to mean ‘is a bijection’. Since Section 5.2 discusses an approach using diagrams, the solution here uses a slightly different approach.

Solution There are no elements $x \in A$ or $y \in A$ such that $p(x) = 1$ or $p(y) = 5$, so $p$ is not a surjection. Inspecting the middle column shows that, for every $z \in B$ there is a $w \in B$ such that $q(z) = w$, so $q$ is a surjection. Similarly (third column), for every $m \in A$ there is a $k \in C$ such that $r(k) = m$, so $r$ is also a surjection.

Inspecting the first column shows that $p(b) = p(d) = 4$ even though $b \neq d$; in words, for two distinct elements of $A$ the function $p$ takes the same value; hence $p$ is not an injection. Similarly (third column) $r(1) = r(4) = a$ although $1 \neq 4$, so $r$ is not an injection.

On the other hand, inspecting the middle column shows that, if $i, j \in B$ and $i \neq j$ then $q(i) \neq q(j)$; in words, for any two distinct elements of $B$ the function $q$ takes distinct values, which is another way of saying that $q$ is an injection.
To be a bijection, a function must be both an injection and a surjection, and the previous results show that $p$ is neither, that $r$ is a surjection but not an injection, and that $q$ is both. It follows that only $q$ is a bijection.
5.2.2 Which of the following functions from \(\mathbb{N}\) to \(\mathbb{N}\) are surjections, which of them are injections, and which of them are bijections?

\[
f(n) = n^3, \quad g(n) = n + 3, \quad h(n) = \begin{cases} 
  n + 1 & \text{if } n \text{ is odd;} \\
  n - 1 & \text{if } n \text{ is even;}
\end{cases}
\]

**Discussion** It is important to realise that whether or not a function is an injection, a surjection or a bijection depends on three things:

- the set \(X\) of possible values of the variable;
- the set \(Y\) of possible values of the function;
- the rule or formula by which the function is defined;

and not just on the third of these. For people familiar with the integers (\(\mathbb{Z}\) - see Chapter 7) or with the real numbers (\(\mathbb{R}\)) and/or the complex numbers (\(\mathbb{C}\)) there are further remarks about this point following the solution. The basic facts about complex numbers can be found at several web pages, including:

- [http://www.ucl.ac.uk/Mathematics/geomath/level2/complex/cn3.html](http://www.ucl.ac.uk/Mathematics/geomath/level2/complex/cn3.html)
- [http://www.clarku.edu/~djoyce/complex/](http://www.clarku.edu/~djoyce/complex/)
- [http://www.bath.ac.uk/~ma1scr/maths1.html](http://www.bath.ac.uk/~ma1scr/maths1.html)
- [http://students.bath.ac.uk/ma1mij/The%20Basics.html](http://students.bath.ac.uk/ma1mij/The%20Basics.html)
- [http://www.maths.abdn.ac.uk/~igc/tch/index/eg1006/notes/node32.html](http://www.maths.abdn.ac.uk/~igc/tch/index/eg1006/notes/node32.html)

and a web search will offer many further references.

**Solution** Suppose that \(m, n \in \mathbb{N}\) and \(f(m) = f(n)\). That means that \(m^3 = n^3\), and hence

\[
0 = m^3 - n^3 = (m - n)(m^2 + mn + n^2).
\]
Since \( m, n \in \mathbb{N} \) it is clear that \((m^2 + mn + n^2) \neq 0\), and the product of two non-zero elements of \( \mathbb{N} \) is a non-zero element of \( \mathbb{N} \), so it must be the case that \( m - n = 0 \), i.e. \( m = n \). Hence \( f \) is an injection. On the other hand, there is no \( k \in \mathbb{N} \) such that \( k^3 = 2 \), since the first few values of \( f(n) \) are \( f(1) = 1, f(2) = 8, f(3) = 27, \ldots \), so \( f \) is not a surjection.

Since \( g(n) = n + 3 \) for all \( n \in \mathbb{N} \) the first few values of \( g(n) \) are
\[
g(1) = 4, \ g(2) = 5, \ g(3) = 6, \ldots,
\]
so there is no \( k \in \mathbb{N} \) such that \( g(k) = 1 \); hence \( g \) is not a surjection. But suppose that \( x, y \in \mathbb{N} \) and \( g(x) = g(y) \); that means \( x + 3 = y + 3 \), and hence (working in \( \mathbb{Z} \)) \( x = (x + 3) - 3 = (y + 3) - 3 = y \), so that \( g \) is an injection.

Suppose that \( m \in \mathbb{N} \) and \( m \) is odd; then \( m + 1 \) is even and \( m+1 \in \mathbb{N} \); also \( h(m+1) = (m+1) - 1 = m \). Alternatively, suppose that \( k \in \mathbb{N} \) and \( k \) is even; then \( k - 1 \) is odd and \( k - 1 \in \mathbb{N} \); also \( h(k - 1) = (k - 1) + 1 = k \). Since every element of \( \mathbb{N} \) is either even or odd, \( h \) is a surjection.

Now suppose that \( u, v \in \mathbb{N} \) and \( h(u) = h(v) \). If this common value is even it follows from the definition of \( h(u) \) that \( u, v \) are odd and that \( h(u) = u + 1 = v + 1 = h(v) \), so that \( u = (u + 1) - 1 = (v + 1) - 1 = v \). A similar argument, if the common value is odd, shows that then too \( u = v \). It follows that \( h \) is also an injection.

Summarising, \( f \) and \( g \) are both injections but not surjections, while \( h \) is both. The only bijection is therefore \( h \).

Remarks The formula \( f(n) = n^3 \) was just found to define a function from \( \mathbb{N} \) to \( \mathbb{N} \) that is an injection but not a surjection.

However, the formula \( F(x) = x^3 \) is, effectively, the same as the one defining \( f \), but if \( x \) is allowed to take real number values in \( \mathbb{R} \) then that \( F : \mathbb{R} \to \mathbb{R} \) is both an injection and a surjection, hence a bijection. (The reason is that every real number \( y \in \mathbb{R} \) has a unique cube root.)

On the other hand, if \( x \) is allowed to take complex number values in \( \mathbb{C} \) then that \( F : \mathbb{C} \to \mathbb{C} \) is a surjection but not an injection, because every complex number \( w \) has at least one cube root, but usually has three: for example (using ‘polar coordinates’): \( 1^3 = (e^{2\pi i/3})^3 = (e^{4\pi i/3})^3 = 1 \), so that \( 1 \) has (at least) three distinct
complex cube roots. (In fact, it has exactly three. Every non-zero complex number has exactly 3 cube roots, while 0 has a unique \( n \)th root - itself - for any \( n \in \mathbb{N} \).) It follows that \( F : \mathbb{C} \rightarrow \mathbb{C} \) is not an injection, and therefore not a bijection.

The function \( F \), whether from \( \mathbb{R} \) to \( \mathbb{R} \) or from \( \mathbb{C} \) to \( \mathbb{C} \), illustrates that whether or not a function is an injection and/or a surjection is closely related to the problem of counting solutions to equations, which often crops up in several areas of pure mathematics.

The formula \( g(n) = n + 3 \) can be extended to give a function from the integers \( \mathbb{Z} \) to \( \mathbb{Z} \), which turns out to be a bijection, unlike the \( g \) in this question. (It is useful to verify the details of the appropriate proofs.)
Solutions to Exercises in *Discrete Mathematics*


5.3 Composition of functions

5.3.1 The functions $s$ and $t$ from $\mathbb{N}$ to $\mathbb{N}$ are defined by

$$s(x) = x + 1, \quad t(x) = 2x \quad (x \in \mathbb{N}).$$

Show that $ts$ and $st$ are different functions.

*Discussion* The best way of showing that two functions $f$ and $g$ are different is to find a value $x$ of the variable such that $f(x) \neq g(x)$; it is sometimes not sufficient to find distinct rules or formulae for the two functions, because in some circumstances (apparently) distinct rules or formulae can take the same values. For example, for $n \in \mathbb{N}$ the formulae

$$f(n) = (n + 5)^2 \quad \text{and} \quad g(n) = (n + 2)^2 + 3 \times (2n + 7)$$

define functions which take identical values throughout $\mathbb{N}$, and are therefore the same, yet it takes a little work to check that the different formulae do have that property.

Similar remarks apply to the functions:

$$(\sin x)^2 \quad \text{and} \quad [1 - (\cos x)^2],$$

where verifying that they take the same values for all $x \in \mathbb{R}$ is, in effect, proving Pythagoras’ Theorem.

*Solution* If $x \in \mathbb{N}$ then $ts(x) = t(s(x)) = t(x + 1) = 2x + 2$ and $st(x) = s(t(x)) = s(2x) = 2x + 1$, so $ts(1) = 4 \neq st(1) = 3$; it follows that $ts \neq st$.

*Remark* In this case, it is ‘obvious’ that the two formulae just found yield different functions, but in other cases it can be much harder to tell, until suitable specific values are substituted into them.
Solutions to Exercises in *Discrete Mathematics*


5.3.2 Let $X = \{1, 2, 3, 4, 5\}$ and let $f : X \to X$ be the function defined by

$$f(1) = 2, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 4, \quad f(5) = 4.$$  

Show that $ff = f$. Find another function $g \neq f$ such that $gf = f$ and $fg = f$.

**Solution** From the definition,

$$(ff)(1) = f(f(1)) = f(2) = 2 = f(1),$$

$$(ff)(2) = f(f(2)) = f(2) = 2 = f(2),$$

$$(ff)(3) = f(f(3)) = f(4) = 4 = f(3),$$

$$(ff)(4) = f(f(4)) = f(4) = 4 = f(4),$$

and

$$(ff)(5) = f(f(5)) = f(4) = 4 = f(5);$$

all values of the variable have been checked, so $ff = f$.

The values taken by $f$ are 2 and 4, so to satisfy the condition $gf = f$ we must have $g(2) = 2$ and $g(4) = 4$. The other condition is that $fg = f$, so we must have $f(g(1)) = f(1) = 2$, which means that $g(1) = 1$ or $g(1) = 2$. Since we want $g \neq f$, pick $g(1) = 1$. It is then possible to pick $g(3) = f(3) = 4$ and $g(5) = f(5) = 5$, while not contradicting any of the three conditions that $g$ is required to satisfy. Thus:

$$g(1) = 1, \quad g(2) = 2, \quad g(3) = 4, \quad g(4) = 4, \quad g(5) = 4$$

defines a suitable $g$.

**Remark** There are other possibilities for $g$, of which the simplest is $g = \text{the identity function}$. 

13
5.4 Bijections and inverse functions

5.4.1 Construct bijections $s, t : S \to S$, where $S$ is the set $\{1, 2, 3\}$, having the following properties. ($i : S \to S$ is the identity function.)

(i) $ss = i$, $s \neq i$. (ii) $ttt = i$, $t \neq i$.

Discussion There are three possibilities for $s$, and two possibilities for $t$; the topic is discussed further at Ex. 2 and in Section 10.6 and in Chapters 20 and 21.

Solution If $s$ is defined by: $s(1) = 2$, $s(2) = 1$ and (therefore) $s(3) = 3$ then $ss = i$ but $s \neq i$. If $t$ is defined by $t(1) = 2$, $t(2) = 3$ and (therefore) $t(3) = 1$ then $ttt = i$ but $t \neq i$. 
5.4.2 Let $S$ be as in Ex. 1. How many different bijections $f$ from $S$ to $S$ are there, and how many of them satisfy $f = f^{-1}$?

**Discussion** Further references for this material are mentioned at Ex. 1.

**Solution** There are six possible bijections from $S$ to $S$, of which two are at Ex. 1 and the following is a list of the others:

- $i$, where $i(1) = 1$, $i(2) = 2$, $i(3) = 3$;
- $s_2$, where $s_2(1) = 1$, $s_2(2) = 3$, $s_2(3) = 2$;
- $s_3$, where $s_3(1) = 3$, $s_3(2) = 2$, $s_3(3) = 1$;
- $t_2$, where $t_2(1) = 3$, $t_2(2) = 1$, $t_2(3) = 2$.

It is easy to check that $t$ and $t_2$ are not equal to their own inverses: in fact $t^{-1} = t_2$ and $t_2^{-1} = t$. The other four bijections are equal to their own inverses.

**Remark** In the above solutions, $i$ is the identity function from $S$ to $S$, which (for any set $S$) satisfies: $i = i^2 = i^3 = \ldots$; $s_2$ and $s_3$ are functions which (like $s$ from Ex. 1) satisfy:

$$s_2s_2 = i \quad \text{but} \quad s_2 \neq i$$

and

$$s_3s_3 = i \quad \text{but} \quad s_3 \neq i;$$

$t_2$ is a function which (like $t$ from Ex. 1) satisfies:

$$t_2t_2t_2 = i \quad \text{but} \quad t_2 \neq i$$

(and, in fact, $t_2t_2 \neq i$ as well).