3.1 The basic logical operations: not, or, and

3.1.1 Construct the truth table for the statement \((\neg p) \land q\).

Discussion Beginners will probably find it helpful to include a column for \((\neg p)\) in their answer, and to show the changes in the values of \(p\) and \(q\) in a systematic fashion, as in the two following solutions.

Solution The table is:

\[
\begin{array}{ccc}
p & q & (\neg p) & (\neg p) \land q \\
T & T & F & F \\
T & F & F & F \\
F & T & T & T \\
F & F & T & F \\
\end{array}
\]
3.1.2 There are eight ways of assigning truth values to three statements $p, q, r$, as in the table on the right. Use this table to determine the corresponding truth values for the statement $(p \land q) \lor r$.

*Note* Here the ‘eight ways’ are shown in the first three columns of the Solution.

**Solution** The completed table is:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$p \land q$</th>
<th>$(p \land q) \lor r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
3.2 Logical equivalence

3.2.1 By constructing the truth table, show that $\neg(\neg p)$ is logically equivalent to $p$.

Solution The completed table is:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$\neg(\neg p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Since the first and last columns are identical, $\neg(\neg p)$ is logically equivalent to $p$. 
Solutions to Exercises in Discrete Mathematics

3.2.2 By constructing truth tables, show that \( \neg(p \land q) \) and \( (\neg p) \lor (\neg q) \) are logically equivalent.

Discussion As in Ex. 3.1.1 and Ex. 3.1.2, it is often helpful to include columns which show the truth values of ‘sub-expressions’, such as the \( (p \land q) \) which occurs inside \( \neg(p \land q) \). The extent to which this will be found helpful will largely depend on experience levels; in the following table the \( \neg p \) and \( \neg q \) columns are easily dispensed with.

Solution The table is:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \land q )</th>
<th>( \neg(p \land q) )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( (\neg p) \lor (\neg q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since the fourth and seventh columns are identical, the two expressions are logically equivalent.
3.3 if - then

3.3.1 Which of the following statements are True, and which of them are False?

- if 256 is a perfect square then 1024 is a perfect square
- if 250 is a perfect square then 1000 is a perfect square
- if 257 is a prime then 1028 is a prime

Discussion In each case, the second number is four times the first number. Each statement is of form ‘if \( p \) then \( q \)’, so we are concerned with whether this implication is true, rather than with the truth or otherwise of the opening ‘\( p \)’ statements. Prime numbers were defined in Ex. 1.2.4:

\[ p \in \mathbb{N} \text{ is prime if } p \geq 2 \text{ and, whenever } m \in \mathbb{N} \text{ and } m \text{ divides } p, \text{ then } m = 1 \text{ or } m = p. \]

(Prime numbers are further discussed in Sections 8.5 and 8.5.)

Solution Since \( 1024 = 4 \times 256 = 2^2 \times 256 \), if 256 is a perfect square then 1024 has to be one as well. For ‘if 256 is a perfect square’ means ‘if 256 = \( n^2 \) for some \( n \in \mathbb{N} \)’, and certainly, if 256 = \( n^2 \) for some \( n \in \mathbb{N} \) then \( 1024 = (2n)^2 \), so 1024 is also a perfect square. Hence the first ‘if - then’ statement is True.

A virtually identical argument shows that, if 250 = \( m^2 \) for some \( m \in \mathbb{N} \) then \( 1000 = (2m)^2 \), so 1000 is also a perfect square. Hence the second ‘if - then’ statement is also True (even though 250 is not a perfect square).

On the other hand, if \( x \in \mathbb{N} \) is prime then \( 4x \) is never prime, because it has ‘non-trivial’ factors 4 and \( x \) as well as 1 and itself. Accordingly, the final ‘if - then’ statement is False.
Remark: These results, about ‘if-then’ statements, illustrate general points which it is important to understand. Thus $256 = 16^2$ and $1024 = 32^2$ are both perfect squares, but neither 250 nor 1000 is: an implication can be True even when the opening ‘$p$’ statement is false. Similarly, the opening ‘$p$’ statement that ‘257 is prime’ is true, but the third implication is nevertheless False.
3.3.2 Construct the truth table for the statement \((\neg p) \lor q\), and deduce that it is logically equivalent to \(p \Rightarrow q\). Verify this fact for each of the three statements in the previous exercise.

Discussion The formal definition of ‘\(\Rightarrow\)’ is in Section 3.3, but is repeated in the Solution (as the final column) for comparison. As previously mentioned, with experience it may be possible to dispense with selected columns.

Solution The completed table is:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(\neg p)</th>
<th>((\neg p) \lor q)</th>
<th>(p \Rightarrow q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since the fourth and fifth columns are identical, the two expressions are logically equivalent. The truth values for the three statements of Ex. 3.3.1, treating each as being of form ‘if \(p\) then \(q\)’, are shown (in order) in the following table:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(p \Rightarrow q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

These results confirm the conclusions reached at Ex. 3.3.1.
3.4 The converse statement

3.4.1 Which of the statements

- if $a$ and $b$ are even numbers then $a + b$ is an even number
- if $a + b$ is an even number then $a$ and $b$ are even numbers

is True and which is False? Justify your answers by giving a proof or a counter-example.

Discussion This exercise illustrates the difference between the statements $p \Rightarrow q$ and $q \Rightarrow p$.

Solution If $a$ and $b$ are even numbers then there are other numbers $r, s \in \mathbb{N}$ such that $a = 2r$ and $b = 2s$, so that $a + b = 2r + 2s = 2(r + s)$, and $(r + s) \in \mathbb{N}$, so $a + b$ is even. It follows that the first statement is True.

If $a = 1$ and $b = 1$ then neither $a$ nor $b$ is even, but $a + b = 1 + 1 = 2 = 2 \times 1$, so $a + b$ is even despite the fact that $a$ and $b$ are both odd. From this counter-example it follows that the second statement is False.
3.4.2 Write down the converse of the following statement.

- if $n$ is a multiple of 3 then $n$ is not a multiple of 7

Say whether the original statement and its converse are True or False, and justify your answers.

Discussion The converse of the statement ‘if $p$ then $q$’ is ‘if $q$ then $p$’. To give a counter-example, as below, is a highly effective way of showing that a statement is False.

Solution The converse of the original statement is:

- if $n$ is not a multiple of 7 then $n$ is a multiple of 3

The original statement is False, since $21 = 3 \times 7$ is a multiple of both 3 and 7. The converse statement is False, since $10 = 2 \times 5$ is not a multiple of either 3 or 7.
3.5 The contrapositive statement

3.5.1 Write down the contrapositive forms of the statements

• if \( n \) is a multiple of 7 then \( n \) is not a multiple of 3
• if \( n \) is a multiple of 12 then \( n \) is a multiple of 4.

Discussion It often helps to think of such statements in a semi-formal way as well as a formal one. For example, the first statement might be thought of as:

• \((n \text{ is a multiple of } 7) \Rightarrow (n \text{ is not a multiple of } 3)\)

or, formally, as:

• \((n = 7r \text{ for some } r \in \mathbb{N}) \Rightarrow (n \neq 3s \text{ for all } s \in \mathbb{N})\)

Solution The contrapositive statements are:

• if \( n \) is a multiple of 3 then \( n \) is not a multiple of 7
• if \( n \) is not a multiple of 4 then \( n \) is not a multiple of 12.
Solutions to Exercises in *Discrete Mathematics*


3.5.2 Are the statements in Exercise 1 True or False?

*Discussion* In collections of exercises in mathematics, it is very common for later questions to be easily solved by the use of solutions to earlier ones (or of solutions in the text). It is also often easier, as here, to check the Truth or Falsity of the contrapositive of a statement; the original, because it is logically equivalent to its contrapositive, will have the same truth value.

*Solution* The first contrapositive was discussed in Ex. 3.4.2, and is known to be False, so the first original statement in Exercise 1 is also False. The second contrapositive is closely related to the first Example in Section 3.3, which contains (in different words and symbols) the True statement

\[ n \in \mathbb{N} \text{ and } 12 \text{ divides } n \text{ then } 4 \text{ divides } n. \]

It follows that the second contrapositive is True, and hence that the second original statement in Exercise 2 is also True.
3.5.3 Are the converses of the statements in Exercise 1 True or False?

Solution The converse statements are:

- if \( n \) is not a multiple of 7 then \( n \) is a multiple of 3
- if \( n \) is not a multiple of 12 then \( n \) is not a multiple of 4.

The first was discussed at Ex. 3.4.2, where it was identified as being False. The second statement is False, since 8 (and 4 itself!) are multiples of 4 but not multiples of 12.
3.6 Universal and existential quantifiers

3.6.1 Which of the following universal statements are True and which of them are False? In each case, give a proof or a counter-example.

\[ \forall n \in \mathbb{N} \; n^3 \geq n^2 \]
\[ \forall n \in \mathbb{N} \; n^3 = n^2 \]
\[ \forall n \in \mathbb{N} \; \text{the remainder when } n^2 \text{ is divided by 4 is 0} \]
\[ \forall n \in \mathbb{N} \; \text{the remainder when } n^2 \text{ is divided by 4 is 0 or 1} \]

*Discussion*  As an aide-memoire, it may be useful to know that: the symbol ‘\(\forall\)’, meaning ‘for All’, is an upside-down letter ‘A’, and the symbol ‘\(\exists\)’, meaning ‘there Exists’, is a back-to-front letter ‘E’. We assume (informal) familiarity with the relations \(\leq\) and \(<\) on \(\mathbb{N}\) (and on the full set \(\mathbb{Z}\) of integers): they are discussed in Sections 4.2 and 7.5.

Also note that the final statement, which is True, has important further mathematical applications.

*Solution*  The first statement is True, since \(n^3 - n^2 = n^2 \times (n - 1)\); the factors \(n^2\) and \(n - 1\) are non-negative, and hence their product is, so that \(n^3 \geq n^2\).

The second statement is False, since \(8 = 2^3 \neq 4 = 2^2\).

The third statement is False, since \(1 = 1^2\) leaves remainder 1, not 0, on division by 4.

The fourth statement is True. To see this, note that if \(n\) is even then \(n = 2m\) for some \(m \in \mathbb{N}\), so \(n^2 = 4m^2\) and leaves remainder 0 on division by 4, while if \(n\) is odd then \(n = 2k - 1\) for some \(k \in \mathbb{N}\), so \(n^2 = 4k^2 - 4k + 1\) and leaves remainder 1 on division by 4.
Solutions to Exercises in *Discrete Mathematics*


3.6.2 Write out the following statement in mathematical notation, using symbols for the quantifiers.

For every pair of natural numbers $x$ and $y$ such that $x > y$ there is a natural number $z$ such that $x > z > y$.

Show that this statement is False by giving a counter-example.

Discussion There are several (minor) variants of a correct formulation: some mathematicians abbreviate ‘such that’, either as ‘s.t.’ or even just as ‘.’, and some prefer to use the ‘if ... then ...’ formulation while others like to use the ‘arrow’ ‘$\Rightarrow$’. Accordingly, several correct solutions are listed below, and it may be of benefit to study the relationships between them.

Solution Each of the following would be a correct answer:

\[ \forall x, y \in \mathbb{N} \text{ such that } x > y, \exists z \in \mathbb{N} \text{ such that } x > z > y \]
\[ \forall x, y \in \mathbb{N} \text{ s.t. } x > y, \exists z \in \mathbb{N} \text{ s.t. } x > z > y \]
\[ \forall x, y \in \mathbb{N}, x > y \Rightarrow (\exists z \in \mathbb{N} \text{ s.t. } x > z > y) \]

The statement itself is False, since no such $z$ exists in the case that $x = 2$ and $y = 1$. 

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